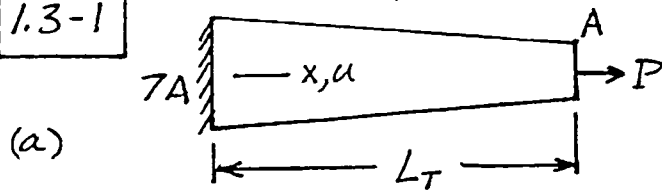


1.3-1



For $0 \leq x \leq L_T$,

$$A_x = A \left(7 - \frac{6x}{L_T} \right)$$

$$\text{and } u = u(x) = \int_0^x \frac{P dx}{A_x E} = \frac{P}{EA} \int_0^x \frac{dx}{7 - \frac{6x}{L_T}}$$

$$u = \frac{P}{EA} \left[- \frac{\ln(7 - 6x/L_T)}{6/L_T} \right]_0^x$$

$$u = \frac{PL_T}{6EA} \left[\ln 7 - \ln \left(7 - \frac{6x}{L_T} \right) \right]$$

	$x = L_T/3$	$x = 2L_T/3$	$x = L_T$
u	0.0556	0.1412	0.3243
	$* PL_T/EA$		

By FEA:

$$\text{At } x = L_T/3, u = \frac{P(L_T/3)}{6EA} = 0.0556 \frac{PL_T}{EA}$$

$$\begin{aligned} \text{At } x = 2L_T/3, u &= 0.0556 \frac{PL_T}{EA} + \frac{P(L_T/3)}{4EA} \\ &= 0.1389 \frac{PL_T}{EA} \end{aligned}$$

$$\begin{aligned} \text{At } x = L_T, u &= 0.1389 \frac{PL_T}{EA} + \frac{P(L_T/3)}{2EA} \\ &= 0.3056 \frac{PL_T}{EA} \end{aligned}$$

(b) FEA stresses at element midpoints:

$$\sigma_{1-2} = \frac{E}{L_T/3} 0.0556 \frac{PL_T}{EA} = 0.167 \frac{P}{A}$$

$$\sigma_{2-3} = \frac{E}{L_T/3} (0.1389 - 0.0556) \frac{PL_T}{EA} = 0.250 \frac{P}{A}$$

$$\sigma_{3-4} = \frac{E}{L_T/3} (0.3056 - 0.1389) \frac{PL_T}{EA} = 0.500 \frac{P}{A}$$

These stresses are exact.

VISIT...

LANZAROTE
Caliente.COM

$$\boxed{1.3-2} \quad u = a_1 + a_2x + a_3y + a_4xy$$

$$\epsilon_x = \frac{\partial u}{\partial x} = a_2 + a_4y$$

Continuity of ϵ_x : might have

$a_2 = a_4 = 0$ in one element and

$a_2 \neq 0, a_4 = 0$ in its neighbor. Thus

$$\boxed{\epsilon_x = 0 \quad \epsilon_x = a_2}$$

and ϵ_x is discontinuous across the shared boundary.

1.3-3 In all parts, substitute coordinates of nodes into Eq.

1.3-5.

$$\begin{aligned} (a) \quad \phi_1 &= a_1 & \therefore a_1 &= \phi_1 \\ \phi_2 &= a_1 + a a_2 & \therefore a_2 &= (\phi_2 - \phi_1)/a \\ \phi_3 &= a_1 + b a_3 & \therefore a_3 &= (\phi_3 - \phi_1)/b \\ \phi &= \phi_1 + \frac{\phi_2 - \phi_1}{a} x + \frac{\phi_3 - \phi_1}{b} y \end{aligned}$$

$$\begin{aligned} (b) \quad \phi_1 &= a_1 + a_2 a & (A) \\ \phi_2 &= a_1 + a_2 a + a_3 b & \left. \begin{array}{l} \\ \end{array} \right\} a_3 = \frac{\phi_2 - \phi_1}{b} & (B) \\ \phi_3 &= a_1 + a_3 b & (C) \end{aligned}$$

$$(B) \& (C) \text{ yield } a_1 = \phi_1 - \phi_2 + \phi_3 \quad (D)$$

$$(A) \& (D) \text{ yield } a_2 = \frac{\phi_2 - \phi_3}{a}$$

$$\phi = (\phi_1 - \phi_2 + \phi_3) + \frac{\phi_2 - \phi_3}{a} x + \frac{\phi_2 - \phi_1}{b} y$$

$$\phi = \left(1 - \frac{y}{b}\right) \phi_1 + \left(-1 + \frac{x}{a} + \frac{y}{b}\right) \phi_2 + \left(1 - \frac{x}{a}\right) \phi_3$$

$$\begin{aligned} (c) \quad \phi_1 &= a_1 - a_2 a & \left. \begin{array}{l} \\ \end{array} \right\} a_1 &= (\phi_1 + \phi_2)/2 \\ \phi_2 &= a_1 + a_2 a & \left. \begin{array}{l} \\ \end{array} \right\} a_2 &= (\phi_2 - \phi_1)/2a \end{aligned}$$

$$\begin{aligned} \phi_3 &= a_1 + a_3 b \\ a_3 &= \frac{\phi_3 - a_1}{b} = \frac{2\phi_3 - \phi_1 - \phi_2}{2b} \end{aligned}$$

$$\phi = \frac{\phi_1 + \phi_2}{2} + \frac{\phi_2 - \phi_1}{2a} x + \frac{2\phi_3 - \phi_1 - \phi_2}{2b} y$$

$$= \left(\frac{1}{2} + \frac{x}{2a} - \frac{y}{2b}\right) \phi_1 + \left(\frac{1}{2} + \frac{x}{2a} - \frac{y}{2b}\right) \phi_2 + \frac{y}{b} \phi_3$$

$$(d) \quad \phi_1 = a_1 \quad \therefore a_1 = \phi_1$$

$$\begin{aligned} \phi_2 &= a_1 + a_2 a + a_3 b \\ \phi_3 &= a_1 + a_2 a + a_3 b & \left. \begin{array}{l} \\ \end{array} \right\} \therefore a_2 &= \frac{\phi_2 - \phi_3}{2a} \end{aligned}$$

$$\text{Add: } \phi_2 + \phi_3 = 2\phi_1 + 2a_3 b; \quad a_3 = \frac{\phi_2 + \phi_3 - 2\phi_1}{2b}$$

$$\phi = \phi_1 + \frac{\phi_2 - \phi_3}{2a} x + \frac{\phi_2 + \phi_3 - 2\phi_1}{2b} y$$

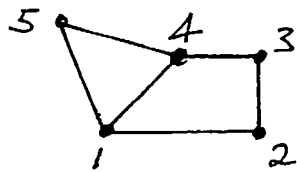
$$\phi = \left(1 - \frac{y}{b}\right) \phi_1 + \left(\frac{x}{2a} + \frac{y}{2b}\right) \phi_2 + \left(-\frac{x}{2a} + \frac{y}{2b}\right) \phi_3$$

1.3-4

Along sides 1-2 and 3-4, y is constant, so ϕ varies linearly with x .
Along side 2-3, x is constant, so ϕ varies linearly with y .

Along side 4-1, $x=y$, so ϕ varies quadratically with x (or with y).

Imagine that adjacent element 1-4-5 is a three-node triangle, for which $\phi = a_1 + a_2x + a_3y$, hence ϕ is linear in x (or y) along each edge. Hence, a gap or overlap is expected along 1-4.

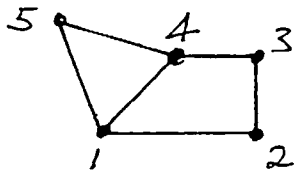


1.3-4

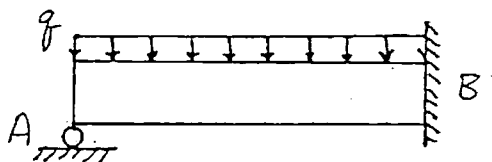
Along sides 1-2 and 3-4, y is constant, so ϕ varies linearly with x .
Along side 2-3, x is constant, so ϕ varies linearly with y .

Along side 4-1, $x=y$, so ϕ varies quadratically with x (or with y).

Imagine that adjacent element 1-4-5 is a three-node triangle, for which $\phi = a_1 + a_2x + a_3y$, hence ϕ is linear in x (or y) along each edge. Hence, a gap or overlap is expected along 1-4.



1.4-1



- Point support at A not really possible.
- Support at A may actually apply a horizontal component of force.
- Load q may not be quite uniform.
- Rigid base at A and rigid wall at B not really possible.
- Plane conditions implied; it is actually 3D to some extent.

1.4-2

- In actual pipe, T_1 and T_2 may vary along the length.
- In a horizontal pipe, temperatures may vary somewhat from top to bottom (e.g. from convection on outside, partial fill of liquid on inside).
- Pipe supports change geometry locally and may add or subtract heat.
- Transients may be important (as from sudden fill with hot water).

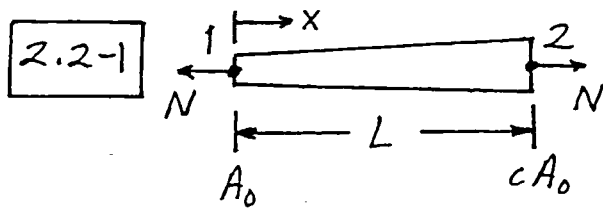
1.4-3

1.4-4

- Omitted from sketch: inspection tunnels (often running normal to the paper), penstocks (running parallel to paper), possible cracks in the concrete, variation normal to paper of the geometry of the rock-concrete interface.
- Considerations: Are loads static or dynamic? If dynamic (e.g. earthquake), what characteristics of the reservoir are important? - level of fill, extent & shape, sloshing of contents?
- Needed: data on geometry (length-wise variation of geometry, including abutments), material properties. Properties of rock include layering, orientation of layers and any fracturing.
- For seepage analysis, need the foregoing data on geometry and material properties. Pertinent properties are permeabilities. Rock ties (excluding cracks) are likely to be orthotropic. Regarding the problem as plane may be an over-simplification.

1.4-5

- In the edge view, loads shown are not collinear. Is this correct? — what realignment or additional moments will appear, perhaps after loading begins?
- Does the rivet fill the hole? Will gaps open or close as load increases? Should friction between plates be credited with carrying appreciable load?
- Is yielding involved?
- Cooling of a hot rivet sets up residual stresses, whose magnitudes influence the load at which yielding begins. What are the residual stresses?



(a) Eq. 2.2-1:

$$[k] = \frac{A_{ave} E}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{(1+c)A_0}{2} \frac{E}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Exact: let δ = elongation

$$\delta = \int_0^L \frac{N dx}{AE} \quad \text{where } A = A_0 + (c-1)\frac{x}{L}A_0$$

$$\delta = \frac{N}{A_0 E} \frac{L}{c-1} \left(1 + (c-1)\frac{x}{L} \right)_0^L = \frac{NL \ln c}{A_0 E (c-1)}$$

$$\text{For } \delta = 1, N = \frac{A_0 E (c-1)}{L \ln c}$$

$$[k] = \frac{A_0 E (c-1)}{L \ln c} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \quad \text{Now let } c=2:$$

Conventional:

$$[k] = 1.5 \frac{A_0 E}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Exact:

$$[k] = 1.443 \frac{A_0 E}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(b) Exact: $\delta = \frac{PL_T}{2A_0 E} \ln 3 = 0.5493 \frac{PL_T}{A_0 E}$

One el.:

$$\delta = \frac{PL_T}{(2A_0)E} = 0.5 \frac{PL_T}{A_0 E} \quad 8.98\% \text{ low}$$

Two els.:

$$\delta = \frac{P(L_T/2)}{A_0 E} \left(\frac{1}{1.5} + \frac{1}{2.5} \right) = 0.5333 \frac{PL_T}{A_0 E}$$

2.91% low

Three els.:

$$\delta = \frac{P(L_T/3)}{A_0 E} \left(\frac{1}{1.333} + \frac{1}{2.000} + \frac{1}{2.667} \right)$$

$$\delta = 0.5417 \frac{PL_T}{A_0 E} \quad 1.39\% \text{ low}$$

Four els.:

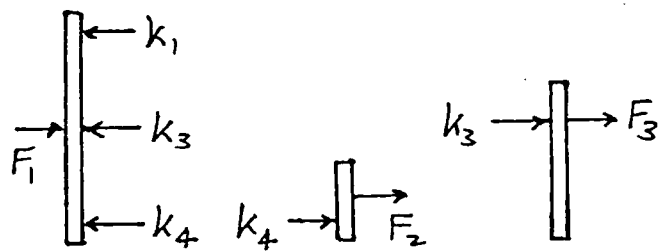
$$\delta = \frac{P(L_T/4)}{A_0 E} \left(\frac{1}{1.25} + \frac{1}{1.75} + \frac{1}{2.25} + \frac{1}{2.75} \right)$$

$$\delta = 0.5449 \frac{PL_T}{A_0 E} \quad 0.81\% \text{ low}$$

2.2-2

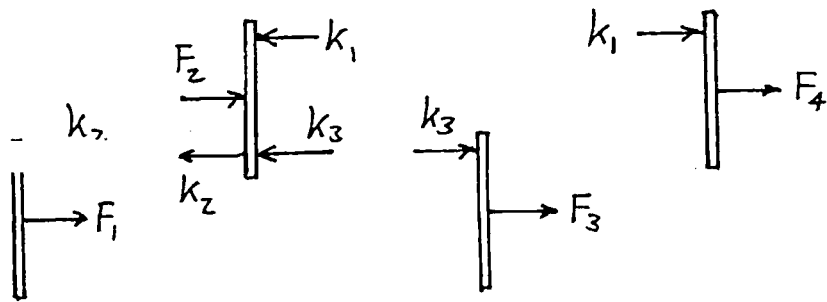
(a)
$$\begin{bmatrix} k_1+k_3+k_4 & -k_4 & -k_3 \\ -k_4 & k_4 & 0 \\ -k_3 & 0 & k_2+k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

Example — column 1:

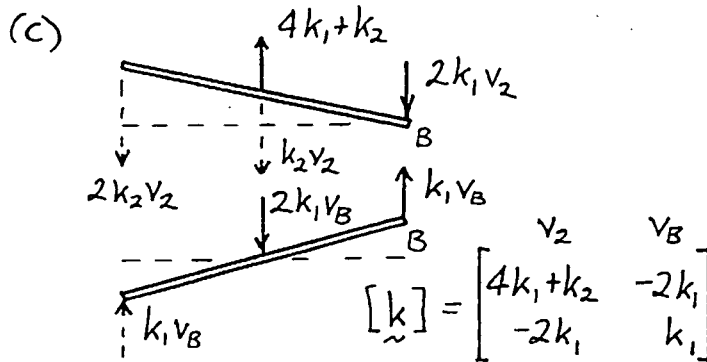
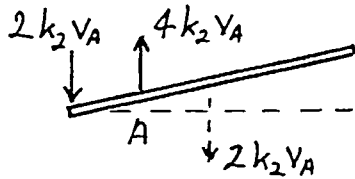
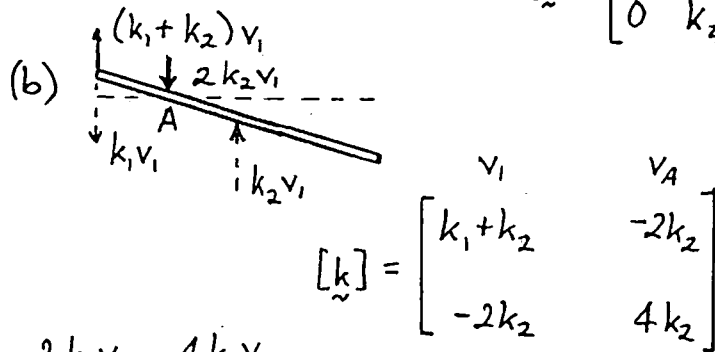
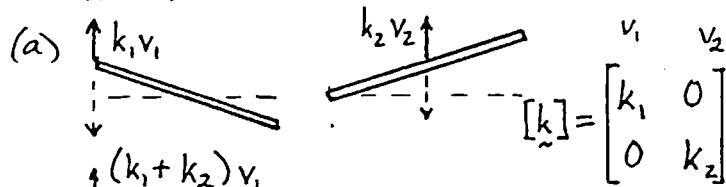


(b)
$$\begin{bmatrix} k_2+k_5+k_6 & -k_2 & -k_6 & 0 \\ -k_2 & k_1+k_2+k_3 & -k_3 & -k_1 \\ -k_6 & -k_3 & k_3+k_4+k_6 & -k_4 \\ 0 & -k_1 & -k_4 & k_1+k_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

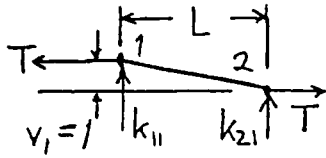
Example — column 2:



2.2-3 Activate d.o.f. in turn to unit value; each time calculate loads that therefore must be applied to the two d.o.f.



2.2-4

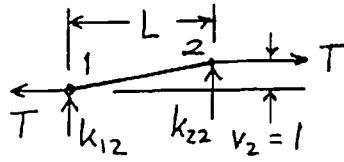


$$\sum M_{node\ 2} = 0$$

$$k_{11}L - T(1) = 0$$

$$k_{11} = \frac{T}{L}$$

$$\sum F_y = 0; \quad k_{21} = -\frac{T}{L}$$



$$\sum M_{node\ 1} = 0$$

$$k_{22}L - T(1) = 0$$

$$k_{22} = \frac{T}{L}$$

$$\sum F_y = 0; \quad k_{12} = -\frac{T}{L}$$

$$[k] = \frac{T}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

2.2-5 Apply Eq. 2.2-9.

$$(a) \quad \frac{k}{t} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ T_2 \\ 200 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ 0 \\ f_3 \end{Bmatrix}$$

2nd eq. gives $T_2 = 100^\circ\text{C}$

1st eq. gives $f_1 = -100k/t$

3rd eq. gives $f_3 = 100k/t$

$$(b) \quad \frac{k}{t} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 400 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ 0 \\ \bar{f} \end{Bmatrix}$$

Unknowns are T_2, T_3, f_1

2nd eq. gives $T_2 = 200 + \frac{T_3}{2}$

Substitute into 3rd equation:

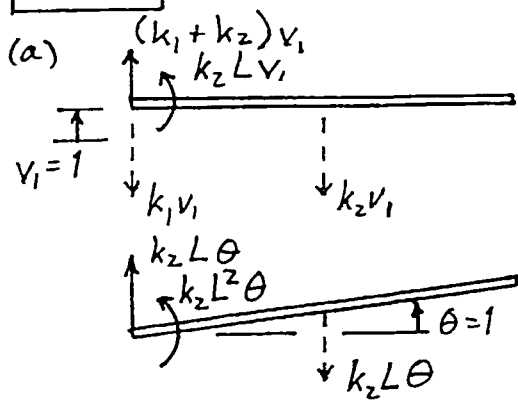
$$\frac{k}{t} \left(-200 - \frac{T_3}{2} + T_3 \right) = \bar{f}$$

$$\therefore T_3 = 400 + 2 \frac{\bar{f}t}{k}$$

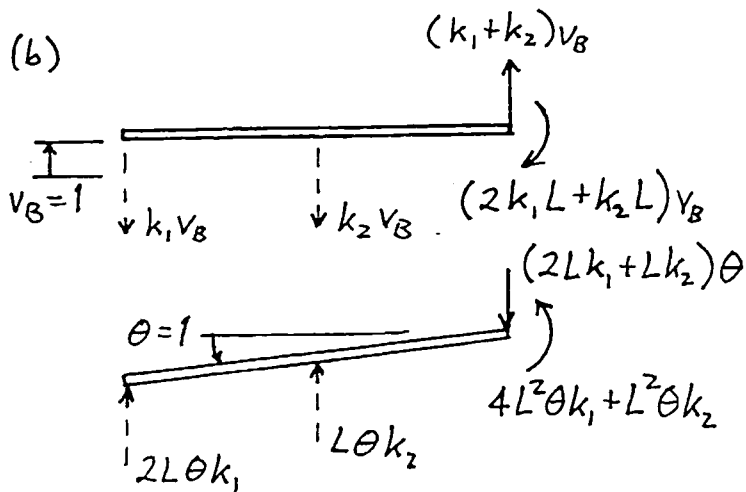
$$\text{Hence } T_2 = 400 + \frac{\bar{f}t}{k}$$

1st eq. gives $f_1 = -\bar{f}$

2.3-1



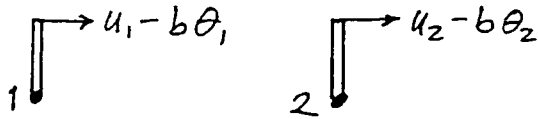
$$[k] = \begin{bmatrix} k_1 + k_2 & k_2 L \\ k_2 L & k_2 L^2 \end{bmatrix}$$



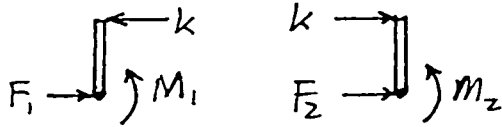
$$[k] = \begin{bmatrix} k_1 + k_2 & -(2k_1 + k_2)L \\ -(2k_1 + k_2)L & (4k_1 + k_2)L^2 \end{bmatrix}$$

2.3-2

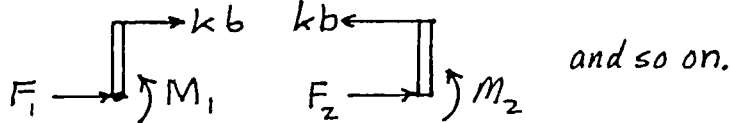
(a) Displacements at spring ends:



Col. 1 of $[k]$: $u_1 = 1, \theta_1 = u_2 = \theta_2 = 0$

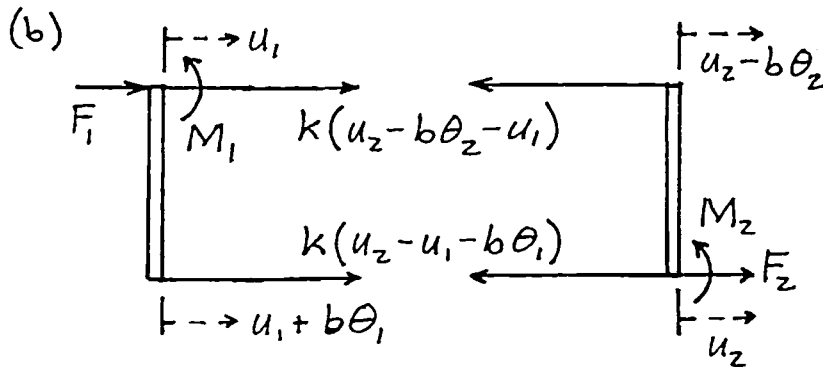


Col. 2 of $[k]$: $u_1 = 0, \theta_1 = 1, u_2 = \theta_2 = 0$



and so on.

$$[k] = \begin{bmatrix} k & -kb & -k & kb \\ -kb & kb^2 & kb & -kb^2 \\ -k & kb & k & -kb \\ kb & -kb^2 & -kb & kb^2 \end{bmatrix}$$



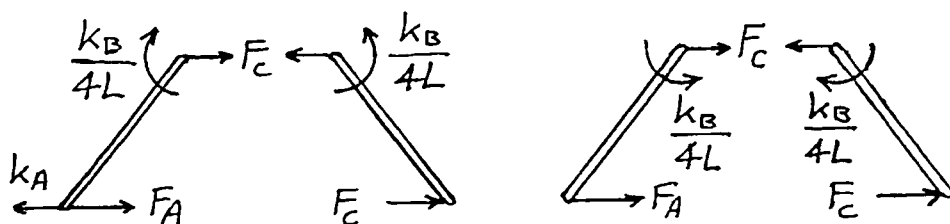
..... $u_1, \theta_1, u_2, \theta_2$ one unit,
in turn.

$$[k] = \begin{bmatrix} 2k & bk & -2k & bk \\ bk & b^2k & -bk & 0 \\ -2k & -bk & 2k & -bk \\ bk & 0 & -bk & b^2k \end{bmatrix}$$

2.3-3

(a) Change of angle at B is $\frac{u_c - u_A}{4L}$

Activate u_A one unit, then u_c one unit.



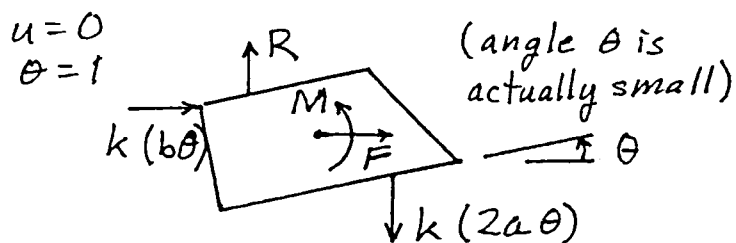
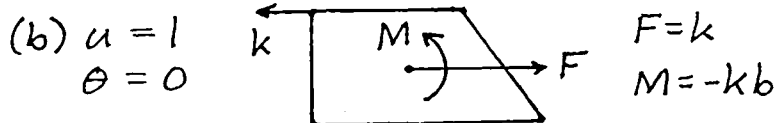
$$4LF_c + \frac{k_B}{4L} = 0; F_c = -\frac{k_B}{16L^2}$$

$$F_A = k_A - F_c = k_A + \frac{k_B}{16L^2}$$

$$F_c = \frac{k_B}{16L^2}$$

$$F_A = -F_c$$

$$[k] = \begin{bmatrix} k_A + \frac{k_B}{16L^2} & -\frac{k_B}{16L^2} \\ -\frac{k_B}{16L^2} & \frac{k_B}{16L^2} \end{bmatrix}$$



$$R = k(2a\theta)$$

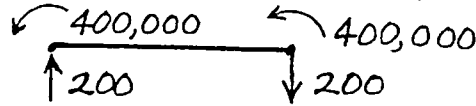
$$M = b[k(b\theta)] + a[R + k(2a\theta)]$$

$$= [b^2 + k(2a^2) + k(2a^2)]\theta$$

$$[k] = \begin{bmatrix} k & -kb \\ -kb & k(4a^2 + b^2) \end{bmatrix}$$

2.3-4

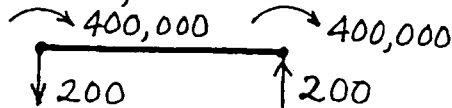
With force units N and length units mm, equilibrium considerations provide the force and moment at the right end.



Thus $k_{11} = 200$, $k_{21} = 400,000$,
 $k_{31} = -200$, $k_{41} = 400,000$.

By the symmetry of $\underline{k}$, we also have the first row of $\underline{k}$.

On physical grounds, we also know that if we lift not the left end but the right end by 1 mm, then



This gives row and column 3 of $\underline{k}$.
 So altogether we know

$$\underline{k} = \begin{bmatrix} 200 & 400,000 & -200 & 400,000 \\ 400,000 & ? & -400,000 & ? \\ -200 & -400,000 & 200 & -400,000 \\ 400,000 & ? & -400,000 & ? \end{bmatrix}$$

2.3-5

(a) Column 2: use Fig. 2.3-1d (beam cantilevered from node 2 with unit rotation at node 1).

$$v_1 = 0 \text{ at node 1} \quad 0 = \frac{k_{12}L^3}{3EI} - \frac{k_{22}L^2}{2EI}$$

$$\theta_{z1} = 1 \text{ at node 1} \quad 1 = -\frac{k_{12}L^2}{2EI} + \frac{k_{22}L}{EI}$$

$$\text{From which } k_{12} = \frac{6EI}{L^2}, \quad k_{22} = \frac{4EI}{L}$$

$$\sum F_y = 0 = k_{12} + k_{32}, \quad k_{32} = -\frac{6EI}{L^2}$$

$$\sum M_{\text{node 2}} = 0 = k_{22} + k_{42} - k_{12}L$$

$$k_{42} = k_{12}L - k_{22} = \frac{2EI}{L}$$

(b) Column 3: use Fig. 2.3-1e (beam cantilevered from node 1 with unit displacement at node 2).

$$v_2 = 1 \text{ at node 2} \quad 1 = \frac{k_{33}L^3}{3EI} + \frac{k_{43}L^2}{2EI}$$

$$\theta_{z2} = 0 \text{ at node 2} \quad 0 = \frac{k_{33}L^2}{2EI} + \frac{k_{43}L}{EI}$$

$$\text{From which } k_{33} = \frac{12EI}{L^3}, \quad k_{43} = -\frac{6EI}{L^2}$$

$$\sum F_y = 0 = k_{13} + k_{33}, \quad k_{13} = -\frac{12EI}{L^3}$$

$$\sum M_{\text{node 1}} = 0 = k_{23} + k_{43} + k_{33}L$$

$$k_{23} = -k_{33}L - k_{43} = -\frac{6EI}{L^2}$$

(c) Column 4: use Fig. 2.3-1f (beam cantilevered from node 1 with unit rotation at node 2).

$$v_2 = 0 \text{ at node 2} \quad 0 = \frac{k_{34}L^3}{3EI} + \frac{k_{44}L^2}{2EI}$$

$$\theta_{z2} = 1 \text{ at node 2} \quad 1 = \frac{k_{34}L^2}{2EI} + \frac{k_{44}L}{EI}$$

$$\text{From which } k_{34} = -\frac{6EI}{L^2}, \quad k_{44} = \frac{4EI}{L}$$

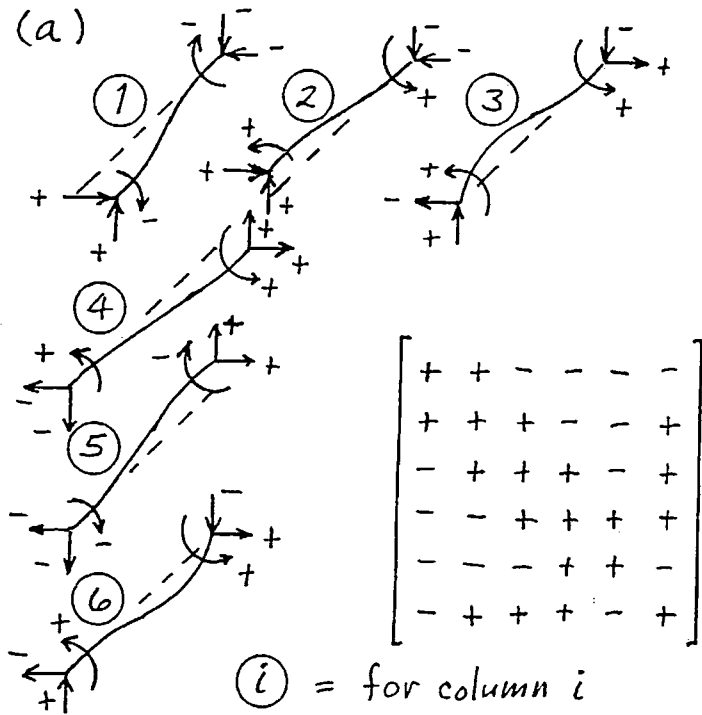
$$\sum F_y = 0 = k_{14} + k_{34}, \quad k_{14} = \frac{6EI}{L^2}$$

$$\sum M_{\text{node 1}} = 0 = k_{24} + k_{44} + k_{34}L$$

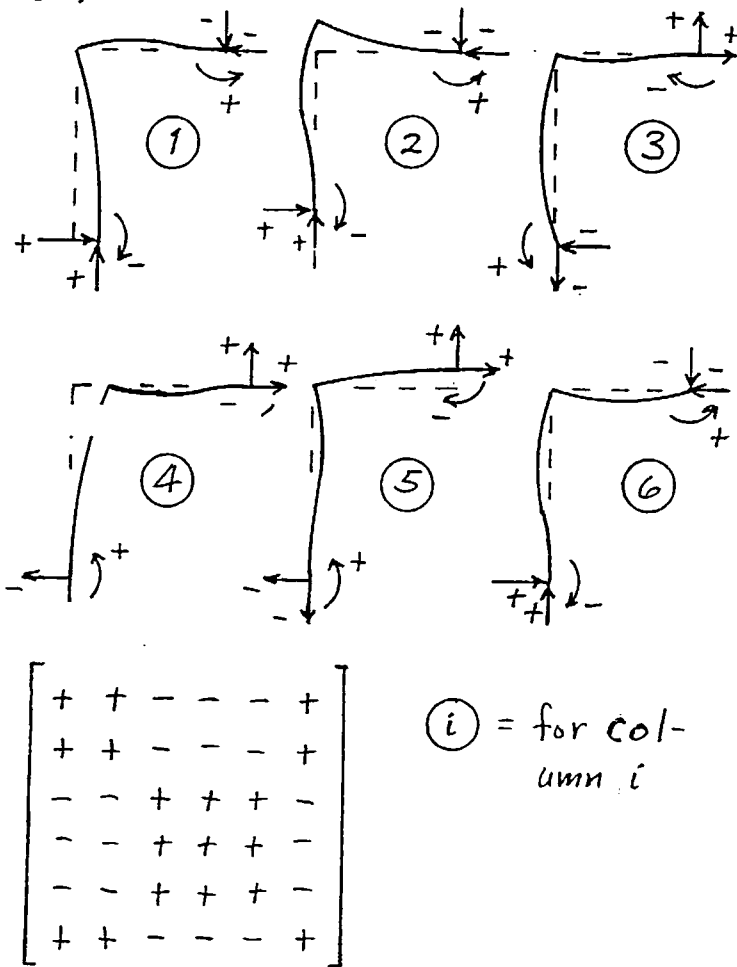
$$k_{24} = -k_{34}L - k_{44} = \frac{2EI}{L}$$

2.3-6

(a)



(b)



2.3-7



(a)

From Eq. 2.3-6, with θ_{z1} and θ_{z2} the active d.o.f.,

$$\frac{EI_z}{(1+\phi_y)L} \begin{bmatrix} 4+\phi_y & 2-\phi_y \\ 2-\phi_y & 4+\phi_y \end{bmatrix} \begin{Bmatrix} \theta_{z1} \\ \theta_{z2} \end{Bmatrix} = \begin{Bmatrix} -M_0 \\ M_0 \end{Bmatrix}$$

From which $\theta_{z1} = -\theta_{z2} = -\frac{M_0 L}{2EI_z}$

(b) $v_2 = \frac{P}{Y_1} = \frac{(1+\phi_y)PL^3}{12EI_z}$

$$\phi_y = \frac{12EI_z k_y}{AGL^2} = \frac{12E[3(4^3)/12]1.2}{3(4)(E/2)L^2} = \frac{38.4}{L^2}$$

$$v_2 = \frac{(1+38.4/L^2)PL^3}{12E[3(4^3)/12]} = \frac{1+38.4/L^2}{192} \frac{PL^3}{E}$$

No trans. shear deformation: $v_2 = \frac{PL^3}{192E} = \bar{v}$

$L=8$: $v_2 = 1.6\bar{v}$; 37.5% from shear

$L=16$: $v_2 = 1.15\bar{v}$; 13.0% from shear

$L=32$: $v_2 = 1.0375\bar{v}$; 3.6% from shear

(c) $v_2 = \frac{(1+\phi_y)PL^3}{12EI_z} = \frac{PL^3}{12EI_z} \left[\frac{AGL^2 + 12EI_z k_y}{AGL^2} \right]$

$$v_2 = \frac{1}{AG} \left[\frac{AGL^2}{12EI_z} + k_y \right]$$

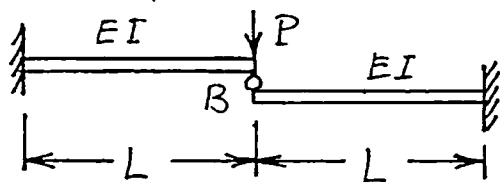
$L \rightarrow \infty$: $v_2 \rightarrow \frac{PL^3}{12EI_z}$

(as in elementary beam theory)

$L \rightarrow 0$: $v_2 \rightarrow \frac{PL}{AG} k_y$

2.3-8

But for torsional stiffness of members, the problem would be the same as the following plane problem:



for which $v = \frac{PL^3}{6EI}$ at B.

For the given problem, note that $\theta_x = \theta_y$ at B. Stiffnesses of the two members add. Including torsional stiffness and letting $\theta_B = \theta_x = \theta_y$ at B, we have

$$\begin{bmatrix} 2 \frac{12EI}{L^3} & 2 \frac{6EI}{L^2} \\ 2 \frac{6EI}{L^2} & 2 \frac{4EI}{L} + 2 \frac{GK}{L} \end{bmatrix} \begin{Bmatrix} v_B \\ \theta_B \end{Bmatrix} = \begin{Bmatrix} -P \\ 0 \end{Bmatrix}$$

Second eq. gives $\theta_B = -\frac{6EI}{(4EI + GK)L} v_B$

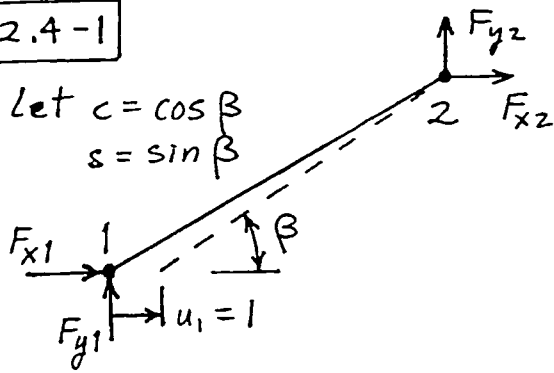
Then first eq. gives

$$v_B = -\frac{PL^3}{24EI} \left[1 - \frac{3}{4 + \frac{GK}{EI}} \right]^{-1}$$

I section: partial restraint of cross-section warping will be present at A, B, and C. Restraint will increase torsional stiffness as compared with foregoing equation and therefore reduce $|v_B|$.

2.4-1

Let $c = \cos \beta$
 $s = \sin \beta$



Shortening $= (1) \cos \beta = c$

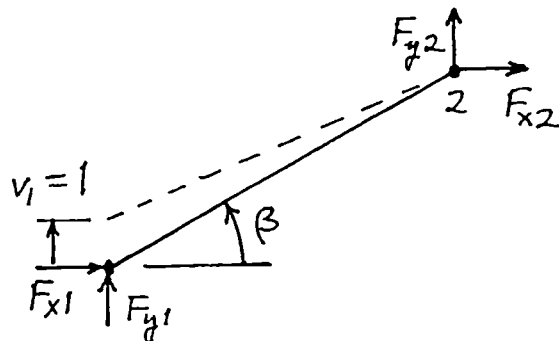
Axial force $= N = \frac{AE}{L} c$

$F_{x1} = -F_{x2} = cN = \frac{AE}{L} c^2$

$F_{y1} = -F_{y2} = sN = \frac{AE}{L} cs$

So column 1 of $[k]$ is

$$\frac{AE}{L} [c^2 \quad cs \quad -c^2 \quad -cs]^T$$



Shortening $= (1) \sin \beta = s$

$N = \frac{AE}{L} s$

$F_{x1} = -F_{x2} = cN = \frac{AE}{L} cs$

$F_{u1} = -F_{u2} = sN = \frac{AE}{L} s^2$

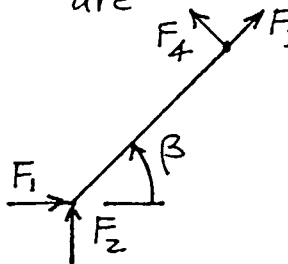
So column 2 of $[k]$ is

$$\frac{AE}{L} [cs \quad s^2 \quad -cs \quad -s^2]^T$$

Similarly for columns 3 and 4 of $[k]$.

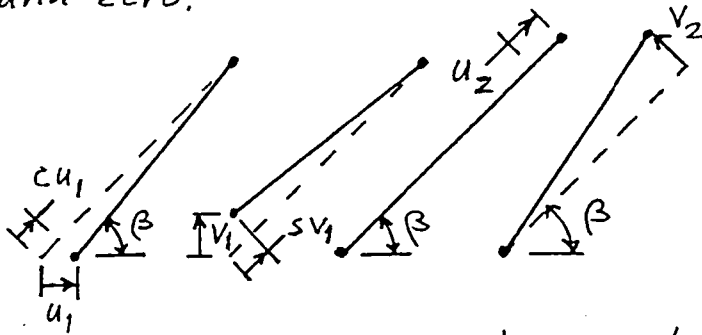
2.4-2

(a) Axial force = $N = \frac{AE}{L} \delta$, where δ is the axial deformation. Nodal forces are



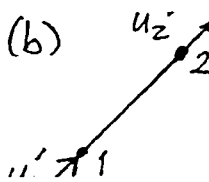
$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{Bmatrix} Nc \\ Ns \\ -N \\ 0 \end{Bmatrix} \quad \begin{matrix} c = \cos \beta \\ s = \sin \beta \end{matrix}$$

Write this column for each of the following cases, whose axial deformations are respectively cu_1 , sv_1 , u_2 , and zero.



with u_1, v_1, u_2, v_2 each unity, we get

$$[k] = \frac{AE}{L} \begin{bmatrix} c^2 & cs & -c & 0 \\ cs & s^2 & -s & 0 \\ -c & -s & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) 

$$[k'] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix} = \underbrace{\begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{[T]} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$[k'] [T] = \frac{AE}{L} \begin{bmatrix} c & s & -1 & 0 \\ -c & -s & 1 & 0 \end{bmatrix}$$

$$[T]^T [k'] [T] = \frac{AE}{L} \begin{bmatrix} c^2 & cs & -c & 0 \\ cs & s^2 & -s & 0 \\ -c & -s & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2.4-3

Let $k = AE/L$

$i = 1, 2$

(a)

D.O.F.: $u_i \quad v_i \quad w_i$

Axial comp.: $l_i u_i \quad m_i v_i \quad n_i w_i$ for bar on x' axis

Axial force: $k l_i u_i \quad k m_i v_i \quad k n_i w_i$

x, y, z force
comps. at
node i ;
 $i = 1, 2$

$$\left\{ \begin{array}{lll} k l_i^2 u_i & k l_i m_i v_i & k l_i n_i w_i \\ k l_i m_i u_i & k m_i^2 v_i & k m_i n_i w_i \\ k l_i n_i u_i & k m_i n_i v_i & k n_i^2 w_i \end{array} \right\} \text{magnitudes}$$

Hence

$$[k] = \frac{AE}{L} \begin{bmatrix} l_i^2 & l_i m_i & l_i n_i & -l_i^2 & -l_i m_i & -l_i n_i \\ l_i m_i & m_i^2 & m_i n_i & -l_i m_i & -m_i^2 & -m_i n_i \\ l_i n_i & m_i n_i & n_i^2 & -l_i n_i & -m_i n_i & -n_i^2 \\ -l_i^2 & -l_i m_i & -l_i n_i & l_i^2 & l_i m_i & l_i n_i \\ -l_i m_i & -m_i^2 & -m_i n_i & -l_i m_i & m_i^2 & m_i n_i \\ -l_i n_i & -m_i n_i & -n_i^2 & l_i n_i & m_i n_i & n_i^2 \end{bmatrix}$$

(b)

$$\begin{aligned} [k'] [T] &= \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} l_i & m_i & n_i & 0 & 0 & 0 \\ 0 & 0 & 0 & l_i & m_i & n_i \end{bmatrix} \\ &= \frac{AE}{L} \begin{bmatrix} l_i & m_i & n_i & -l_i & -m_i & -n_i \\ -l_i & -m_i & -n_i & l_i & m_i & n_i \end{bmatrix} \end{aligned}$$

$$[k] = [T]^T ([k'] [T])$$

Gives same $[k]$ as in part (a).

2.4-4

matrix to transform is $[k'] = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$

$$(a) \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_A \end{Bmatrix} = [T] \begin{Bmatrix} v_1 \\ v_A \end{Bmatrix}$$

$$[T]^T([k'] [T]) = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} k_1 & 0 \\ -k_2 & 2k_2 \end{bmatrix} = \begin{bmatrix} k_1 + k_2 & -2k_2 \\ -2k_2 & 4k_2 \end{bmatrix}$$

$$(b) \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} v_2 \\ v_B \end{Bmatrix} = [T] \begin{Bmatrix} v_2 \\ v_B \end{Bmatrix}$$

$$[T]^T([k'] [T]) = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2k_1 & -k_1 \\ k_2 & 0 \end{bmatrix} = \begin{bmatrix} 4k_1 + k_2 & -2k_1 \\ -2k_1 & k_1 \end{bmatrix}$$

$$(c) \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta \end{Bmatrix} = [T] \begin{Bmatrix} v_1 \\ \theta \end{Bmatrix}$$

$$[T]^T([k'] [T]) = \begin{bmatrix} 1 & 1 \\ 0 & L \end{bmatrix} \begin{bmatrix} k_1 & 0 \\ k_2 & k_2 L \end{bmatrix} = \begin{bmatrix} k_1 + k_2 & k_2 L \\ k_2 L & k_2 L^2 \end{bmatrix}$$

$$(d) \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{bmatrix} 1 & -2L \\ 1 & -L \end{bmatrix} \begin{Bmatrix} v_B \\ \theta \end{Bmatrix} = [T] \begin{Bmatrix} v_B \\ \theta \end{Bmatrix}$$

$$[T]^T([k'] [T]) = \begin{bmatrix} 1 & 1 \\ -2L & -L \end{bmatrix} \begin{bmatrix} k_1 & -2Lk_1 \\ k_2 & -Lk_2 \end{bmatrix} = \begin{bmatrix} k_1 + k_2 & -(2k_1 + k_2)L \\ -(2k_1 + k_2)L & (4k_1 + k_2)L^2 \end{bmatrix}$$

2.5-1

$$[k]_1 \{d\}_1 = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \begin{Bmatrix} d_2 \\ d_3 \\ d_1 \end{Bmatrix} \quad \begin{array}{l} d_2 \rightarrow D_1 \\ d_3 \rightarrow D_4 \\ d_1 \rightarrow D_2 \end{array}$$

$$[k]_2 \{d\}_2 = \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{bmatrix} \begin{Bmatrix} d_2 \\ d_3 \\ d_1 \end{Bmatrix} \quad \begin{array}{l} d_2 \rightarrow D_4 \\ d_3 \rightarrow D_3 \\ d_1 \rightarrow D_2 \end{array}$$

Reorder, expand, and add.

$$\underbrace{\begin{pmatrix} \begin{bmatrix} a_1 & a_3 & 0 & a_4 \\ a_7 & a_9 & 0 & a_8 \\ 0 & 0 & 0 & 0 \\ a_4 & a_6 & 0 & a_5 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & b_9 & b_8 & b_7 \\ 0 & b_6 & b_5 & b_4 \\ 0 & b_3 & b_2 & b_1 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix} \end{pmatrix}}_{[K]_2}$$

2.5-2

$[K]_{\sim}$:

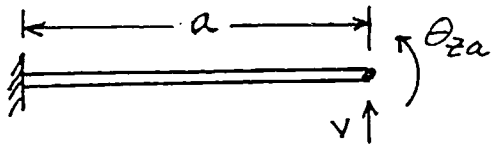
$$\begin{bmatrix} A,C & C & A & C & C \\ C & C & & C & C \\ A & & A,B & B & B \\ C & C & B & B,C & B,C \\ C & C & B & B,C & B,C \end{bmatrix}$$

$\{R\}_{\sim}$:

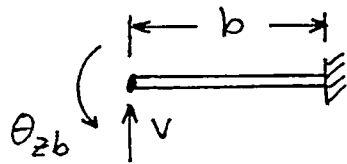
$$\left\{ \begin{array}{l} A,C \\ C \\ A,B \\ B,C \\ B,C \end{array} \right\}$$

2.5-3

No rotational connection at the hinge —
retain two θ_z d.o.f. there, so $\{\underline{D}\} = [v \quad \theta_{za} \quad \theta_{zb}]^T$



$$[k]_a = EI_z \begin{bmatrix} 12/a^3 & -6/a^2 & 0 \\ -6/a^2 & 4/a & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} v \\ \theta_{za} \\ \theta_{zb} \end{matrix}$$

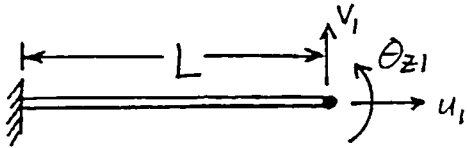


$$[k]_b = EI_z \begin{bmatrix} 12/b^3 & 0 & -6/b^2 \\ 0 & 0 & 0 \\ -6/b^2 & 0 & 4/b \end{bmatrix} \begin{matrix} v \\ \theta_{za} \\ \theta_{zb} \end{matrix}$$

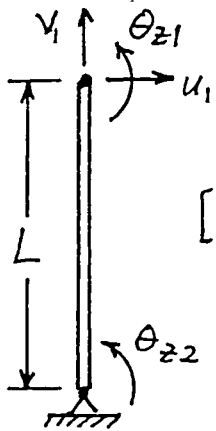
$$[K] = EI_z \begin{bmatrix} 12/a^3 + 12/b^3 & -6/a^2 & -6/b^2 \\ -6/a^2 & 4/a & 0 \\ -6/b^2 & 0 & 4/b \end{bmatrix} \begin{matrix} v \\ \theta_{za} \\ \theta_{zb} \end{matrix}$$

2.5-4

$$a = AE/L \quad b = EI/L^3$$



$$[k] = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & 12b & -6bL & 0 \\ 0 & -6bL & 4bL^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \theta_{z1} \\ \theta_{z2} \end{bmatrix}$$

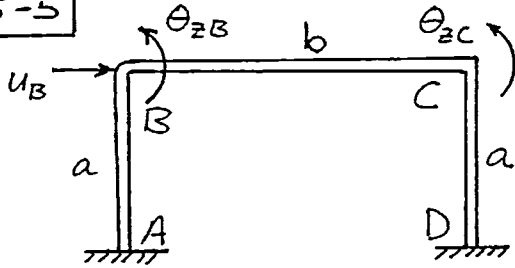


$$[k] = \begin{bmatrix} 12b & 0 & 6bL & 6bL \\ 0 & a & 0 & 0 \\ 6bL & 0 & 4bL^2 & 2bL^2 \\ 6bL & 0 & 2bL^2 & 4bL^2 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \theta_{z1} \\ \theta_{z2} \end{bmatrix}$$

Add; get

$$[K] = \begin{bmatrix} a+12b & 0 & 0 & 0 \\ 0 & a+12b & -6bL & 0 \\ 6bL & -6bL & 8bL^2 & 2bL^2 \\ 6bL & 0 & 2bL^2 & 4bL^2 \end{bmatrix}$$

2.5-5



$$\frac{EI_z}{a^3} \begin{bmatrix} 12 & 6a & 0 \\ 6a & 4a^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_B \\ \theta_{zB} \\ \theta_{zC} \end{bmatrix}$$

$$\frac{EI_z}{b^3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4b^2 & 2b^2 \\ 0 & 2b^2 & 4b^2 \end{bmatrix} \begin{bmatrix} u_B \\ \theta_{zB} \\ \theta_{zC} \end{bmatrix}$$

$$\frac{EI_z}{a^3} \begin{bmatrix} 12 & 0 & 6a \\ 0 & 0 & 0 \\ 6a & 0 & 4a^2 \end{bmatrix} \begin{bmatrix} u_B \\ \theta_{zB} \\ \theta_{zC} \end{bmatrix}$$

$[K] = \text{sum of the above}$

$$[K] = EI_z \begin{bmatrix} 24/a^3 & 6/a^2 & 6/a^2 \\ 6/a^2 & 4/a + 4/b & 2/b \\ 6/a^2 & 2/b & 4/a + 4/b \end{bmatrix} \begin{bmatrix} u_B \\ \theta_{zB} \\ \theta_{zC} \end{bmatrix}$$

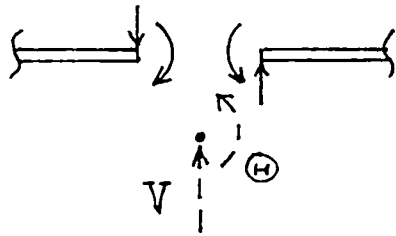
2.5-6

Sign changes in $[k]$ of Eq. 2.3-5:

- In columns 3 and 4 (because directions of d.o.f. are reversed)
- In rows 3 and 4 (because directions of loads are reversed)

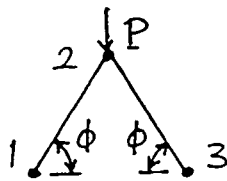
Thus k_{33} and k_{44} end up positive.

Awkward because d.o.f. don't match when elements are assembled.



If V and H are global d.o.f., must change signs again in left-hand element to make its $\{d\}$ match $\{D\}$.

2.5-7



$$c = \cos \phi$$

$$s = \sin \phi$$

Bar 1-2: Eq. 2.4-6,
with $\beta = \phi$

$$[k] = \frac{AE}{L} \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

Bar 2-3: Eq. 2.4-6, with $\beta = -\phi$

$$[k] = \frac{AE}{L} \begin{bmatrix} c^2 & -cs & -c^2 & cs \\ -cs & s^2 & cs & -s^2 \\ -c^2 & cs & c^2 & -cs \\ cs & -s^2 & -cs & s^2 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

Omit rows and columns corresponding to fixed dof. u_1, v_1, u_3, v_3 and sum overlapping terms at node 2.

$$\frac{AE}{L} \begin{bmatrix} 2c^2 & 0 \\ 0 & 2s^2 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \end{Bmatrix} \quad \therefore u_2 = 0$$

$$v_2 = -\frac{PL}{2AEs^2}$$

Member elongation: $e = v_2 \sin \phi = v_2 s$

$$\epsilon = \frac{e}{L} = -\frac{P}{2AEs} \quad \sigma = E\epsilon = -\frac{P}{2As}$$

as $\phi \rightarrow 0$

(according to linear small-displacement theory)

2.6-1

$$(a) \begin{bmatrix} u_1 & v_1 & \theta_{z1} \\ u_2 & v_2 & \theta_{z2} \end{bmatrix} =$$

$$\begin{bmatrix} c_1 & c_2 & c_3 \\ c_1 & c_2 + Lc_3 & c_3 \end{bmatrix}$$

where c_1, c_2, c_3 are constants,
and c_3 must be small.

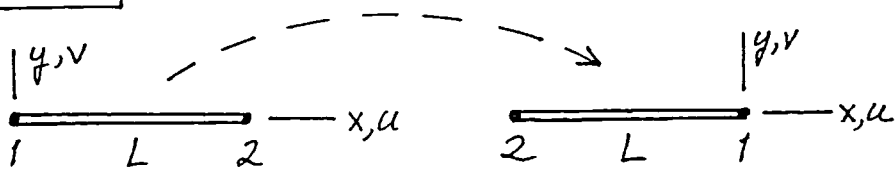
(b)

$$\begin{bmatrix} u_1 & v_1 & w_1 & \theta_{x1} & \theta_{y1} & \theta_{z1} \\ u_2 & v_2 & w_2 & \theta_{x2} & \theta_{y2} & \theta_{z2} \end{bmatrix} =$$

$$\begin{bmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ c_1 & c_2 + Lc_6 & c_3 - Lc_5 & c_4 & c_5 & c_6 \end{bmatrix}$$

where constants c_5 and c_6 must be small.

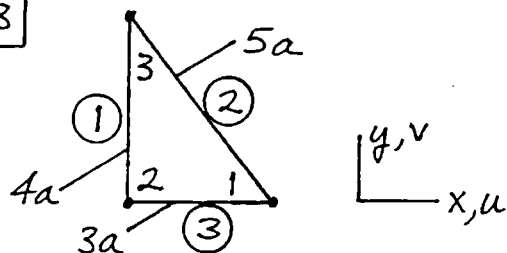
2.6-2



$$\begin{bmatrix} u_1 & v_1 & \theta_{z1} & u_2 & v_2 & \theta_{z2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \pi & -2L & 0 & \pi \end{bmatrix}$$

$[\underline{k}]\{\underline{d}\} \neq \{0\}$ because $[\underline{k}]$ is based on the original (undeformed, undisplaced) geometry, and so is a good approximation only if deformations and rotational displacements are small.

2.6-3



$$(a) \{ \underline{D} \}_1 = c_1 \begin{bmatrix} -3 & 4 & -3 & 4 & -3 & 4 \end{bmatrix}^T$$

where c_1 is a constant

$$(b) \{ \underline{D} \}_2 = c_2 \begin{bmatrix} 4a & 3a & 4a & 0 & 0 & 0 \end{bmatrix}^T$$

where c_2 is a small constant

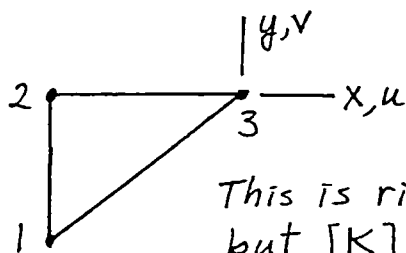
$$(c) \{ \underline{D} \}_3 = c_3 \begin{bmatrix} 4a & 0 & 4a & -3a & 0 & -3a \end{bmatrix}^T$$

where c_3 is a small constant

(d) Yes: there are no values of the c_i such that the $\{ \underline{D} \}_i$ sum to zero.

$$(e) \{ \underline{D} \} = \begin{bmatrix} -7 & -3 & -4 & 0 & 0 & -4 \end{bmatrix}^T \text{ gives}$$

the displaced structure as



This is rigid-body displacement,
but $[\underline{K}] \{ \underline{D} \} \neq \{ \underline{Q} \}$ for the reason
stated in Problem 2.6-2.

2.6-4

(a) We get row-sums (or column-sums, due to symmetry of $[k]$) from $[k]\{d\}$ with $\{d\} = [1 \ 1 \ 1 \ 1 \ 1 \ 1]^T$, but this $\{d\}$ is not rigid-body motion.

(b) $\{d\} = [c_1 \ c_2 \ c_3 \ c_1 - Rc_3 \ c_2 + Rc_3 \ c_3]^T$

where the c_i are constants and c_3 must be small.

(c) $c_1 \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} \quad c_2 \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix} \quad c_3 \begin{Bmatrix} 0 \\ 0 \\ 1 \\ -R \\ R \\ 1 \end{Bmatrix}$

where c_3 is a small constant.

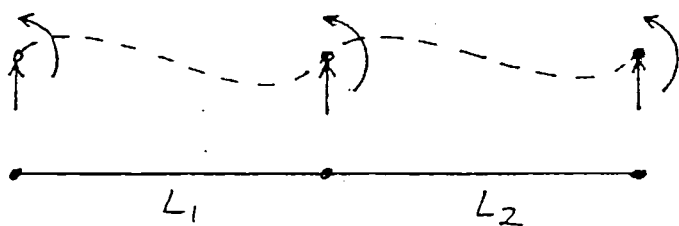
The latter vector is rigid-body rotation about the node at $x=y=0$.

2.6-5

(a) We get row-sums (or column-sums, due to symmetry of $[K]$) from $[K]\{\underline{D}\}$ with $\{\underline{D}\} = [1 \ 1 \ 1 \ 1 \ 1 \ 1]^T$, but this $\{\underline{D}\}$ is not rigid-body motion.

(b) $\{\underline{D}\} = [c_1 \ c_2 \ c_1 + L_1 c_2 \ c_2 \ c_1 + (L_1 + L_2)c_2 \ c_2]^T$ where the c_i are constants and c_2 is small.

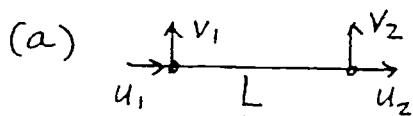
(c)



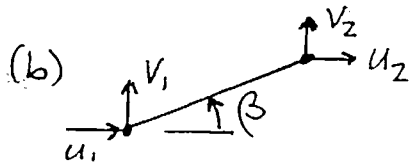
Direction of middle force arrow is upward, as shown, if $L_1 > L_2$.

2.6-6 $\{\underline{d}\} = [u_1 \ v_1 \ u_2 \ v_2]^T$

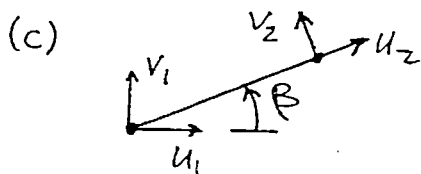
Let c_1, c_2, c_3 be constants, with c_3 a small rotation.



$$\{\underline{d}\} = [c_1 \ c_2 \ c_1 \ c_2 + Lc_3]^T$$



$$\{\underline{d}\} = [c_1 \ c_2 \ c_1 - c_3 L \sin \beta \ c_2 + c_3 L \cos \beta]^T$$

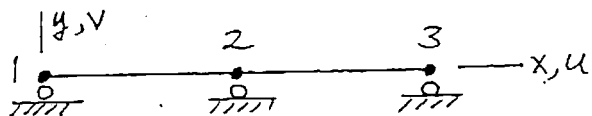


$$\{\underline{d}\} = [c_1 \ c_2 \ c_1 \cos \beta + c_2 \sin \beta \ -c_1 \sin \beta + c_2 \cos \beta + c_3 L]^T$$

2.6-7

(a) With n_R restraints it is possible to overconstrain one motion while at the same time underconstraining another. Examples, using beam elements:

2D: $n_R = 3$, d.o.f. u, v, θ_z at each node



Only v_1, v_2, v_3 constrained

3D:



$n_R = 6$, d.o.f. $u, v, w, \theta_x, \theta_y, \theta_z$ at each node.

Only θ_x (or θ_y, θ_z) restrained at each node.

(b) Always restrain u, v, θ_z at node 1.

Also restrain:

(1) w, θ_x, θ_y at node 1, or

(2) w, θ_x, θ_y at node 2, or

(3) w, θ_x, θ_y at node 3, or

(4) w at nodes 1 and 2, θ_x at node 2, or

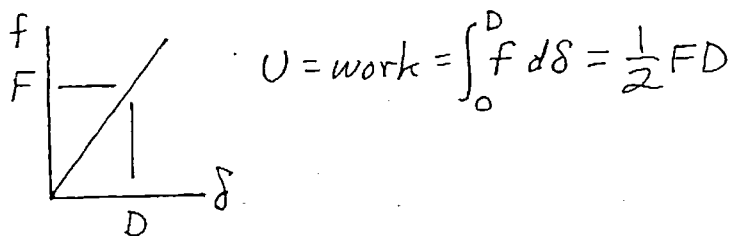
(5) w at nodes 1 and 2, θ_x at node 3, or

(6) w at node 2, θ_x at node 1, θ_y at node 3,

Etc.

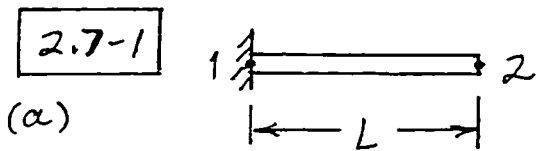
Strain energy in a linearly elastic structure due to gradually applied loads $\{\underline{R}\}$ is $\{\underline{D}\}^T \{\underline{R}\} / 2$, e.g. for a single load F ,

2.6-8



But $\{\underline{R}\} = [\underline{K}] \{\underline{D}\}$, so

$$U = \frac{1}{2} \{\underline{D}\}^T [\underline{K}] \{\underline{D}\}$$



From Eq. 2.3-5,

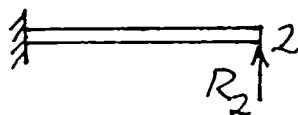
$$EI_z \begin{bmatrix} 12/L^3 & -6/L^2 \\ -6/L^2 & 4/L \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_{z2} \end{Bmatrix} = \begin{Bmatrix} R_2 \\ M_2 \end{Bmatrix} \quad (A)$$

Set $v_2 = \bar{v}_2$

$$EI_z \begin{bmatrix} 1 & 0 \\ 0 & 4/L \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_{z2} \end{Bmatrix} = \begin{Bmatrix} EI_z \bar{v}_2 \\ 6EI_z \bar{v}_2 / L^2 \end{Bmatrix} \text{ gives } \theta_{z2} = \frac{3\bar{v}_2}{2L}, v_2 = \bar{v}_2$$

Eq. (A) then gives $R_2 = EI_z \left[\frac{12}{L^3} \bar{v}_2 - \frac{6}{L^2} \frac{3\bar{v}_2}{2L} \right] = \frac{3EI_z \bar{v}_2}{L^3}$

Beam theory:



$$v_2 = \frac{R_2 L^3}{3EI_z}, \quad \theta_{z2} = \frac{R_2 L^2}{2EI_z}$$

$$\theta_{z2} = \frac{L^2}{2EI_z} \frac{3EI_z}{L^3} v_2 = \frac{3v_2}{2L}$$

$$R_2 = \frac{2EI_z}{L^2} \theta_{z2} = \frac{3EI_z}{L^3} v_2$$

(b) From Eq. 2.3-5,

$$\frac{2EI_z}{L} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} \theta_{z1} \\ \theta_{z2} \end{Bmatrix} = \begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix} \quad (B)$$

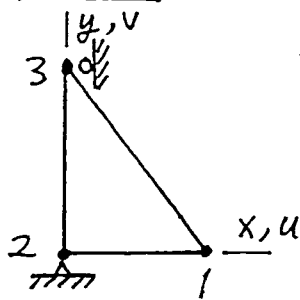
Set $\theta_{z1} = \bar{\theta}_{z1}$

$$\frac{2EI_z}{L} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} \theta_{z1} \\ \theta_{z2} \end{Bmatrix} = \begin{Bmatrix} 2EI_z \bar{\theta}_{z1} / L \\ -2EI_z \bar{\theta}_{z1} / L \end{Bmatrix} \text{ gives } \theta_{z1} = \bar{\theta}_{z1}, \theta_{z2} = -\frac{\bar{\theta}_{z1}}{2}$$

Eq. (B) then gives $M_1 = \frac{2EI_z}{L} \left[2\bar{\theta}_{z1} - \frac{\bar{\theta}_{z1}}{2} \right] = \frac{3EI_z}{L} \bar{\theta}_{z1}$

These results appear in tables of beam deflections.

2.7-2



For $k_1 = k_2 = k_3 = k$, $P = 0$, Eq. 2.7-8 is

$$k \begin{bmatrix} 1.36 & -0.48 & 0.48 \\ -0.48 & 0.64 & -0.64 \\ 0.48 & -0.64 & 1.64 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (A)$$

Impose $u_1 = c, v_1 = 0$ by method of Eq. 2.7-6

$$k \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1.64 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} ck \\ 0 \\ -0.48ck \end{Bmatrix} \quad \therefore \begin{Bmatrix} u_1 \\ v_1 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} c \\ 0 \\ -0.2927c \end{Bmatrix}$$

Use the latter vector on the left-hand side of Eq. (A).

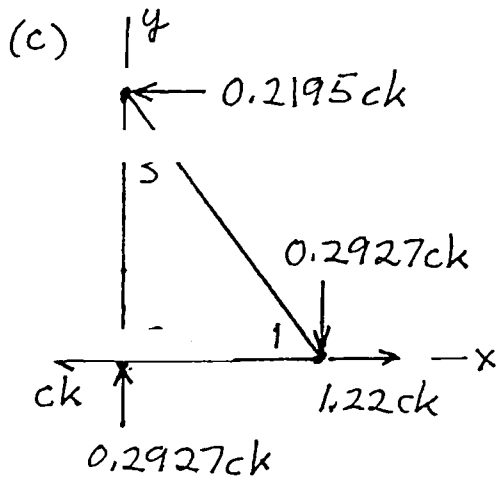
$$k \begin{bmatrix} 1.36 & -0.48 & 0.48 \\ -0.48 & 0.64 & -0.64 \\ 0.48 & -0.64 & 1.64 \end{bmatrix} \begin{Bmatrix} c \\ 0 \\ -0.2927c \end{Bmatrix} = \begin{Bmatrix} 1.22ck \\ -0.2927ck \\ 0 \end{Bmatrix} \quad \begin{matrix} \leftarrow F_{x1} \\ \leftarrow F_{y1} \end{matrix}$$

(b) Return to Eq. 2.5-10 with $k_1 = k_2 = k_3 = k$ and

$$\{D\} = [c \quad 0 \quad 0 \quad 0 \quad 0 \quad -0.2927c]^T$$

Hence

$$[K]\{D\} = \{R\} = ck [1.22 \quad -0.2927 \quad -1 \quad 0.2927 \quad -0.2195 \quad 0]^T$$

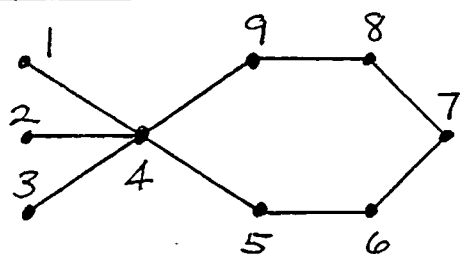


$$\sum F_x = 0 \quad \checkmark$$

$$\sum F_y = 0 \quad \checkmark$$

$$\sum M_2 = (0.2195ck)4 - (0.2927ck)3 = 0 \quad \checkmark$$

2.8-1



x			x						1
	x		x						2
		x	x						3
			x	x				x	4
				x	x				5
					x	x			6
						x	x		7
							x	x	8
								x	9
1	2	3	4	5	6	7	8	9	

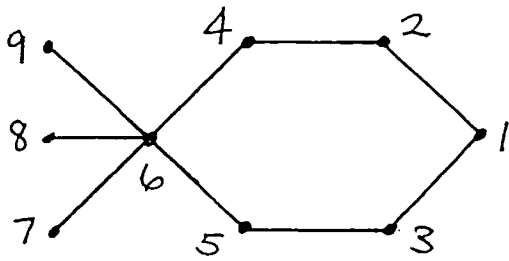
symmetric

$b_{max} = 6$ (in row 4)

$r \approx 1$

$fills = 3$ (in column 9)

2.8-2



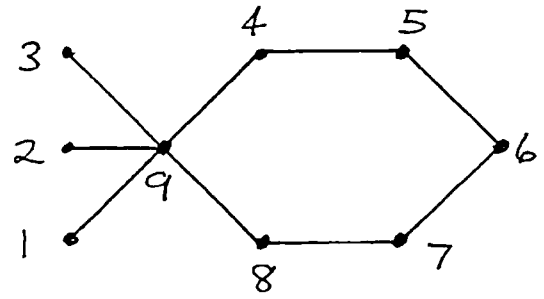
1	2	3	4	5	6	7	8	9	
x	x	x							1
	x		x						2
		x		x					3
			x		x				4
				x	x				5
					x	x	x	x	6
						x			7
							x		8
								x	9

Symm.

$$p = 24$$

$$b_{max} = 4 \text{ (row 6)}$$

$$fills = 6$$



1	2	3	4	5	6	7	8	9	
x								x	1
	x							x	2
		x						x	3
			x	x				x	4
				x	x				5
					x	x			6
						x	x		7
							x	x	8
								x	9

Symm.

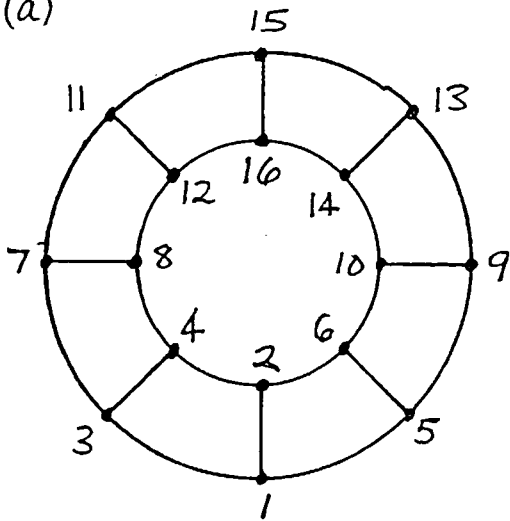
$$p = 21$$

$$b_{max} = 9 \text{ (row 1)}$$

$$fills = 3$$

2.8-3

(a)



(candidate numbering)

$$b_{\max} = 5$$

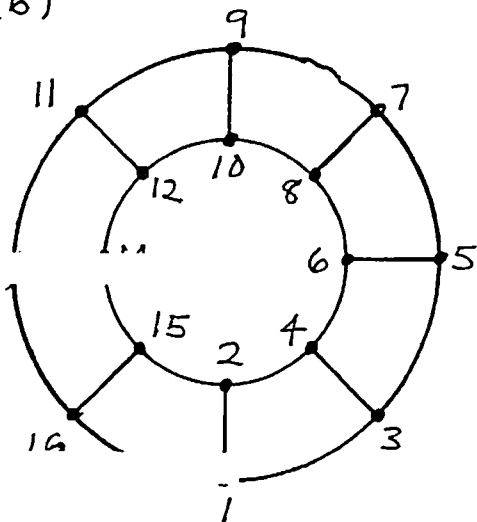
$$p = 12(5) + 3 + 3 + 2 + 1 = 69$$

$$\text{fills} = 29$$

1	2	3	4	5	6	7	8	9	10	12	14	16	
X	X	X		X									1
	X		X		X								2
		X	X			X							3
			X				X						4
				X	X			X					5
					X				X				6
						X	X			X			7
							X				X		8
								X	X			X	9
									X				10
										X	X		11
											X		12
												X	13
													14
													15
													16

symmetric

(b)



(candidate numbering)

$$b_{\max} = 16 \text{ (row 1)}$$

$$p = 12(3) + 14 + 16 = 66$$

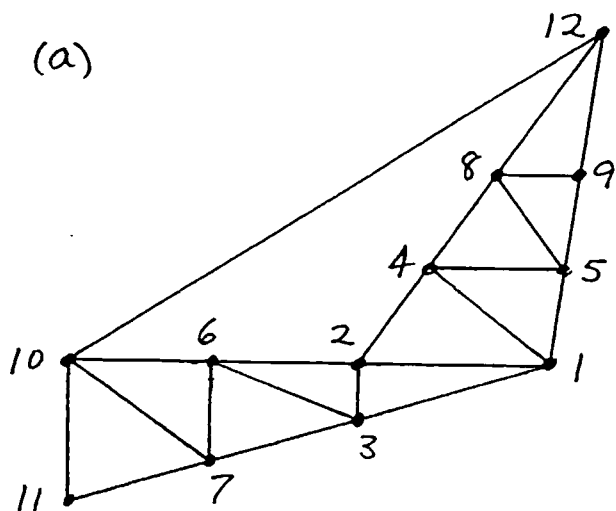
$$\text{fills} = 6 + 11 + 12 = 29$$

1	2	3	4	5	6	7	8	9	10	12	14	16	
X	X	X											1
	X		X								X		2
		X	X	X									3
			X		X								4
				X	X	X							5
					X		X						6
						X	X	X					7
							X		X				8
								X	X	X			9
									X				10
										X	X	X	11
											X		12
												X	13
													14
													15
													16

symmetric

2.8-4

(a)



(candidate numbering)

$$b_{\max} = 5$$

$$p = 8(5) + 4 + 3 + 2 + 1 = 50$$

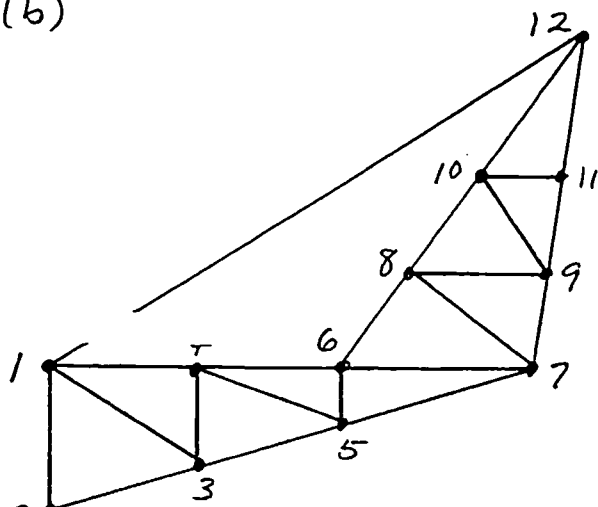
$$\text{fills} = 16$$

1	2	3	4	5	6	7	8	9	10	12		
x	x	x	x	x							1	
	x	x	x		x						2	
		x			x	x					3	
			x	x			x				4	
				x			x	x			5	
					x	x			x		6	
						x			x	x	7	
							x	x		x	8	
								x		x	9	
									x	x	10	
										x	11	
											x	12

symmetric

symmetric

(b)



(candidate numbering)

$$b_{\max} = 12$$

$$p = 8(3) + 4 + 2 + 1 + 12 = 43$$

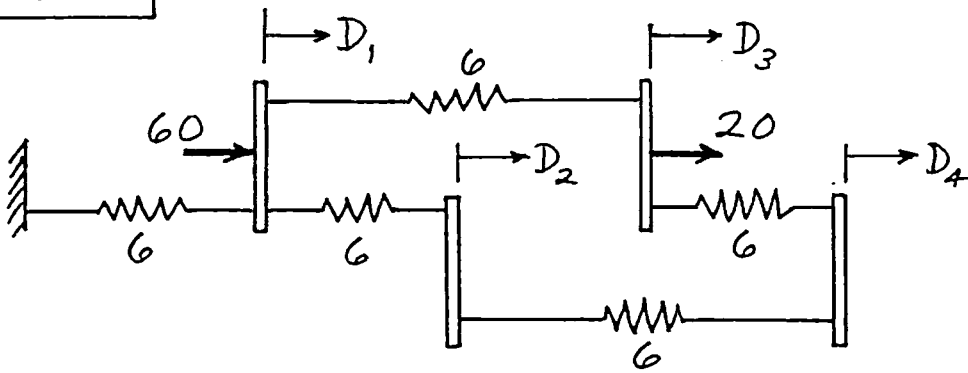
$$\text{fills} = 9$$

1	2	3	4	5	6	7	8	9	10	12		
X	X	X	X							X	1	
	X	X									2	
		X	X	X							3	
			X	X	X						4	
				X	X	X					5	
					X	X	X				6	
						X	X	X			7	
							X	X	X		8	
								X	X	X	9	
									X	X	X	10
										X	X	11
											X	12

symmetric

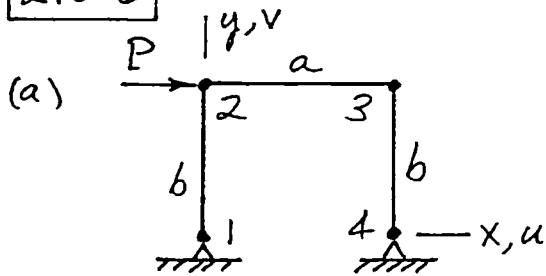
symmetric

2.8-5



Spring k 's and nodal loads shown by numbers.

2.8-6



$$AE \begin{bmatrix} 1/a & 0 & -1/a & 0 \\ 0 & 1/b & 0 & 0 \\ -1/a & 0 & 1/a & 0 \\ 0 & 0 & 0 & 1/b \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

After the first elimination,

$$AE \begin{bmatrix} 1/a & 0 & -1/a & 0 \\ 0 & 1/b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/b \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \\ P \\ 0 \end{Bmatrix}$$

Trouble will appear in the third elimination (the u_3 equation).

(b)

$$\frac{EI_2}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_{21} \\ v_2 \\ \theta_{22} \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

After the first elimination,

$$\frac{EI_2}{L^3} \begin{bmatrix} \dots & 6L & -12 & 6L \\ 0 & L^2 & 0 & -L^2 \\ 0 & 0 & 0 & 0 \\ 0 & -L^2 & 0 & L^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_{21} \\ v_2 \\ \theta_{22} \end{Bmatrix} = \begin{Bmatrix} P \\ -PL/2 \\ P \\ -PL/2 \end{Bmatrix}$$

Trouble will appear in the third elimination (the v_2 equation).

In both cases, trouble appears in the first equation for which restraint becomes necessary to prevent rigid body motion or a mechanism. If no mechanism is possible and d.o.f. include rotations as well as displacements, the offending equation involves the last-numbered node, since full restraint at this node would suffice to prevent rigid body motion.

2.8-7

(a) Given system

$$\begin{bmatrix} 12 & -6 & 0 \\ -6 & 12 & -6 \\ 0 & -6 & 6 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 24 \\ 24 \\ 0 \end{Bmatrix}$$

2nd elimination

$$\begin{bmatrix} 12 & -6 & 0 \\ 0 & 9 & -6 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 24 \\ 36 \\ 24 \end{Bmatrix}$$

1st elimination

$$\begin{bmatrix} 12 & -6 & 0 \\ 0 & 9 & -6 \\ 0 & -6 & 6 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 24 \\ 36 \\ 0 \end{Bmatrix}$$

Back-substitution

$$u_4 = 24/2 = 12$$

$$u_3 = (36 + 6u_4)/9 = (36 + 72)/9 = 12$$

$$u_2 = (24 + 6u_3)/12 = (24 + 72)/12 = 8$$

(b) After the first elimination, $K_{22} = 9$. This is the stiffness seen by u_3 when node 2 is free to move, so members 1 and 2 are in series (for which $k = 3$). Stiffness $k = 3$ adds to stiffness $k = 6$ of member 3. After the second elimination, $K_{33} = 2$. This is the stiffness seen by node 4 when nodes 2 and 3 are freed, so that node 4 sees all three members in series, for which $\frac{1}{k} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$ and hence $k = 2$.

(c) 1st iteration 2nd iteration 3rd iteration 4th iteration

$$u_2 = 2$$

$$u_2 = 3.50$$

$$u_2 = 4.625$$

$$u_2 = 5.469$$

$$u_3 = 3$$

$$u_3 = 5.25$$

$$u_3 = 6.9375$$

$$u_3 = 8.203$$

$$u_4 = 3$$

$$u_4 = 5.25$$

$$u_4 = 6.9375$$

$$u_4 = 8.203$$

2.8-8

(a) Given eqs.

$$k \begin{bmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ F_2 \\ 0 \end{bmatrix}$$

2nd elimination

$$k \begin{bmatrix} 3 & -1 & -1 \\ 0 & 2/3 & -1/3 \\ 0 & 0 & 3/2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ F_2 \\ F_2/2 \end{bmatrix}$$

1st elimination

$$k \begin{bmatrix} 3 & -1 & -1 \\ 0 & 2/3 & -1/3 \\ 0 & -1/3 & 5/3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ F_2 \\ 0 \end{bmatrix}$$

Back substitution

$$\begin{aligned} ku_3 &= \frac{1}{3} F_2, & u_3 &= \frac{F_2}{3k} \\ ku_2 &= \frac{3}{2} \left[F_2 + \frac{1}{3} \frac{F_2}{3} \right], & u_2 &= \frac{5F_2}{3k} \\ ku_1 &= \frac{1}{3} \left[\frac{5F_2}{3} + \frac{F_2}{3} \right], & u_1 &= \frac{2F_2}{3k} \end{aligned}$$

(b) Given $\{R\}$ 1st elimination 2nd elimination

$$\begin{bmatrix} F_1 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} F_1 \\ F_1/3 \\ F_1/3 \end{bmatrix} \longrightarrow \begin{bmatrix} F_1 \\ F_1/3 \\ F_1/3 + F_1/6 \end{bmatrix} = F_1 \begin{bmatrix} 1 \\ 1/3 \\ 1/2 \end{bmatrix}$$

Back substitution

$$\begin{aligned} ku_3 &= \frac{2}{3} \frac{F_1}{2} = \frac{F_1}{3} \\ ku_2 &= \frac{3}{2} \left[\frac{F_1}{3} + \frac{1}{3} \frac{F_1}{3} \right] = \frac{2F_1}{3} \\ ku_1 &= \frac{1}{2} \left[F_1 + \frac{2F_1}{3} + \frac{F_1}{3} \right] = \frac{2F_1}{3} \end{aligned}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \frac{F_1}{k} \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

2,8-9 Given eqs.

1st elimination

$$(a) \quad k \begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ F_2 \\ 0 \\ 0 \end{bmatrix}, \quad k \begin{bmatrix} 3 & -1 & -1 & 0 \\ 0 & 8/3 & -4/3 & -1 \\ 0 & -4/3 & 8/3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ F_2 \\ 0 \\ 0 \end{bmatrix}$$

2nd elimination

$$k \begin{bmatrix} 3 & -1 & -1 & 0 \\ 0 & 8/3 & -4/3 & -1 \\ 0 & 0 & 2 & -3/2 \\ 0 & 0 & -3/2 & 13/8 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ F_2 \\ F_2/2 \\ 3F_2/8 \end{bmatrix}$$

3rd elimination

$$k \begin{bmatrix} 3 & -1 & -1 & 0 \\ 0 & 8/3 & -4/3 & -1 \\ 0 & 0 & 2 & -3/2 \\ 0 & 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ F_2 \\ F_2/2 \\ 3F_2/4 \end{bmatrix}$$

Back substitution

$$ku_4 = \frac{3F_2}{2}$$

$$ku_3 = \frac{1}{2} \left[\frac{F_2}{2} + \frac{3}{2} \frac{3F_2}{2} \right] = \frac{11F_2}{8}$$

$$ku_2 = \frac{3}{8} \left[F_2 + \frac{4}{3} \frac{11F_2}{8} + \frac{3F_2}{2} \right] = \frac{13F_2}{8}$$

$$ku_1 = \frac{1}{3} \left[\frac{13F_2}{8} + \frac{11F_2}{8} \right] = F_2$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \frac{F_2}{k} \begin{bmatrix} 1.000 \\ 1.625 \\ 1.375 \\ 1.500 \end{bmatrix}$$

(b) Given $\{R_i\}$ 1st elim. 2nd elim. 3rd elim.

$$\begin{bmatrix} F_1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} F_1 \\ F_1/3 \\ F_1/3 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} F_1 \\ F_1/3 \\ F_1/3 + F_1/6 \\ F_1/8 \end{bmatrix} \rightarrow \begin{bmatrix} F_1 \\ F_1/3 \\ F_1/2 \\ F_1/8 + 3F_1/8 \end{bmatrix} = F_1 \begin{bmatrix} 1 \\ 1/3 \\ 1/2 \\ 1/2 \end{bmatrix}$$

Back substitution

$$ku_4 = F_1$$

$$ku_3 = \frac{1}{2} \left[\frac{F_1}{2} + \frac{3F_1}{2} \right] = F_1$$

$$ku_2 = \frac{3}{8} \left[\frac{F_1}{3} + \frac{4}{3} F_1 + F_1 \right] = F_1$$

$$ku_1 = \frac{1}{3} [F_1 + F_1 + F_1] = F_1$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \frac{F_1}{k} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

2.8-10

(a)

$$\begin{bmatrix} 1.36 & -0.48 & 0.48 \\ -0.48 & 0.64 & -0.64 \\ 0.48 & -0.64 & 1.64 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P/k \\ 0 \end{Bmatrix}$$

Divide row 1 by 1.36, then mult. by 0.48 & -0.48 resp.; add to rows 2 & 3.

$$\begin{bmatrix} 1 & -0.3529 & 0.3529 \\ 0 & 0.4706 & -0.4706 \\ 0 & -0.4706 & 1.4706 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P/k \\ 0 \end{Bmatrix}$$

Add row 2 to row 3, then divide row 2 by 0.4706.

$$\begin{bmatrix} 1 & -0.3529 & 0.3529 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -2.125P/k \\ -P/k \end{Bmatrix}$$

Back-substitute.

$$v_3 = -P/k$$

$$v_1 = -2.125P/k - P/k = -3.125P/k$$

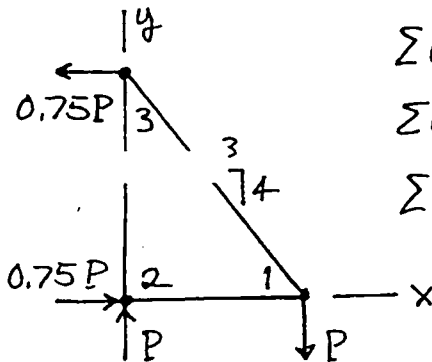
$$u_1 = 0.3529(-3.125P/k) - 0.3529(-P/k) = -0.75P/k$$

(b) Use Eq. 2.5-10

$$P_2 = k \left[-(-0.75 \frac{P}{k}) + 0 \right] = 0.75P$$

$$q_2 = k \left[0 - (-\frac{P}{k}) \right] = P$$

$$P_1 = k \left[-0.36(-0.75 \frac{P}{k}) + 0.48(-3.125 \frac{P}{k}) + 0.36(0) - 0.48(-\frac{P}{k}) \right] = -0.75P$$



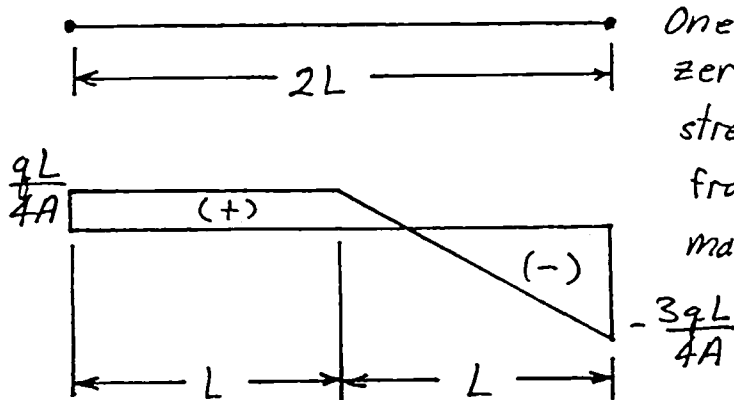
$$\sum F_x = 0 \checkmark$$

$$\sum F_y = 0 \checkmark$$

$$\sum M_2 = 0.75P(4) - P(3) = 0 \checkmark$$

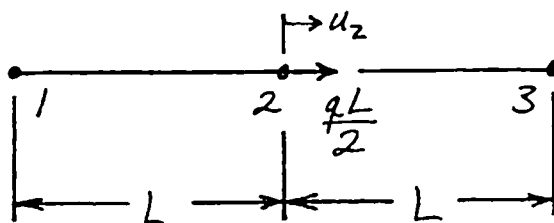
2.9-1

(a)



One element - nodal d.o.f. are zero, so the only contribution to stress is element stress, which from elementary mechanics of materials is as shown.

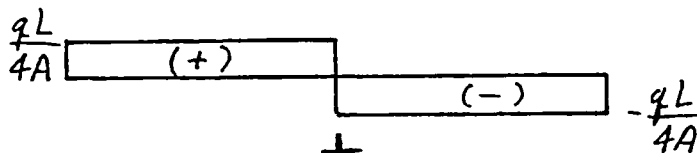
(b)



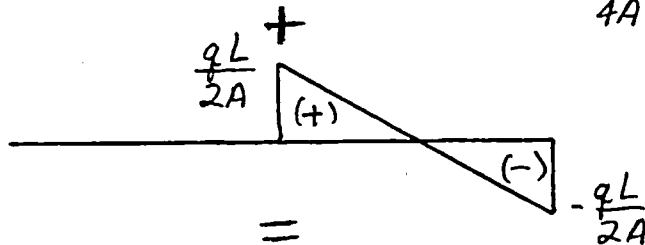
$$u_2 = \frac{qL/2}{2AE/L} = \frac{qL^2}{4AE}$$

$$\epsilon_{1-2} = \frac{u_2}{L}, \sigma_{1-2} = E\epsilon_{1-2} = \frac{qL}{4A}$$

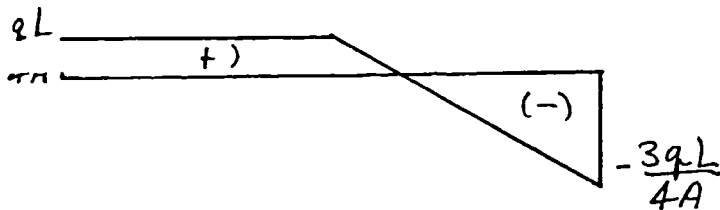
$$\epsilon_{2-3} = -\frac{u_2}{L}, \sigma_{2-3} = E\epsilon_{2-3} = -\frac{qL}{4A}$$



Stress due to u_2

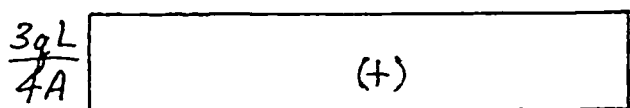
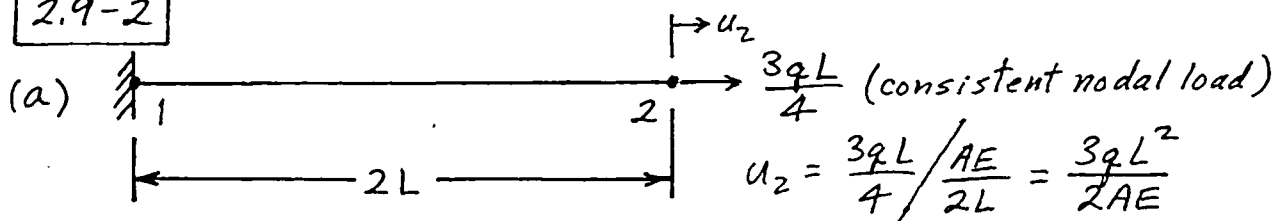


Element stress



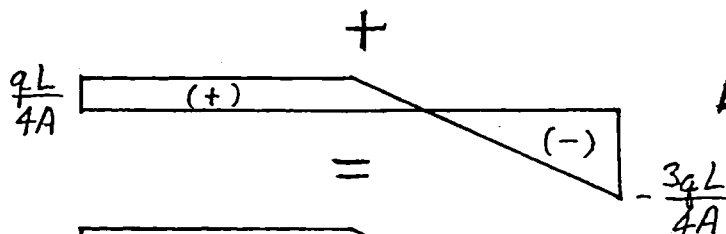
Sum of the two

2.9-2

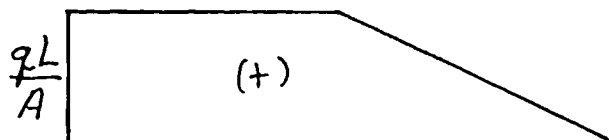


$$\sigma_2 = E \frac{u_2}{2L} = \frac{3qL}{4A}$$

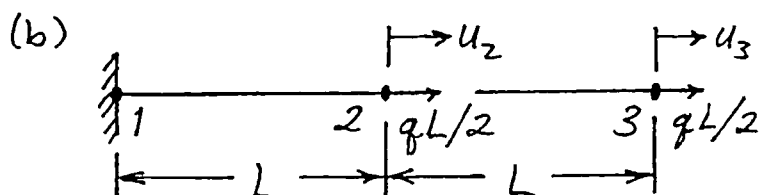
Stress due to u_2



Element stress

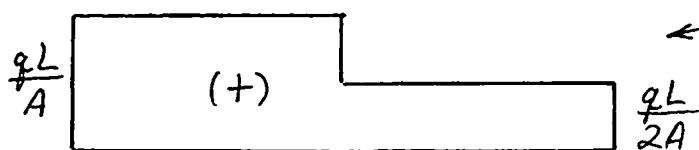


Sum of the two

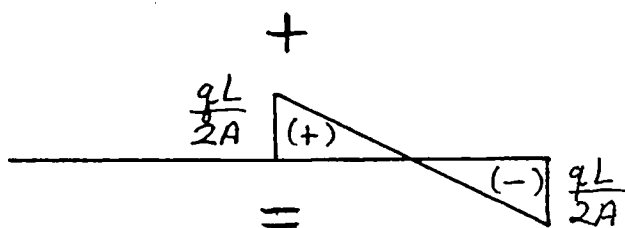


$$u_2 = \frac{qL}{AE/L} = \frac{qL^2}{AE}$$

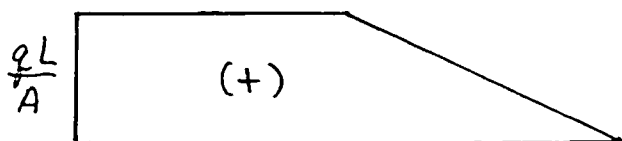
$$u_3 = u_2 + \frac{qL/2}{AE/L} = \frac{3qL^2}{2AE}$$



$$\left\{ \begin{array}{l} \sigma_{1-2} = E \frac{u_2}{L} = \frac{qL}{A} \\ \sigma_{2-3} = E \frac{u_3 - u_2}{L} = \frac{qL}{2A} \end{array} \right.$$

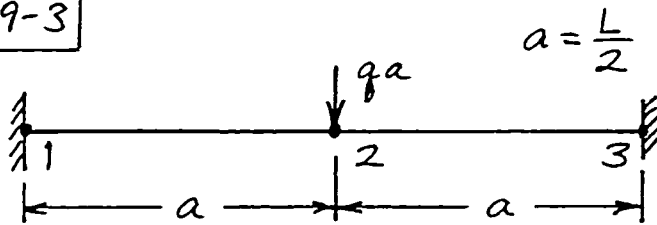


Element stress



Sum of the two

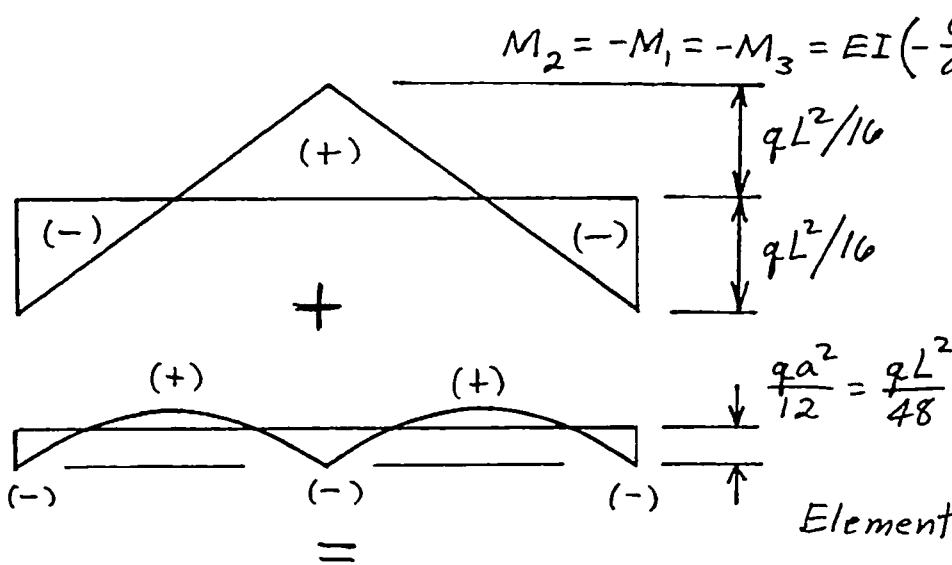
2.9-3



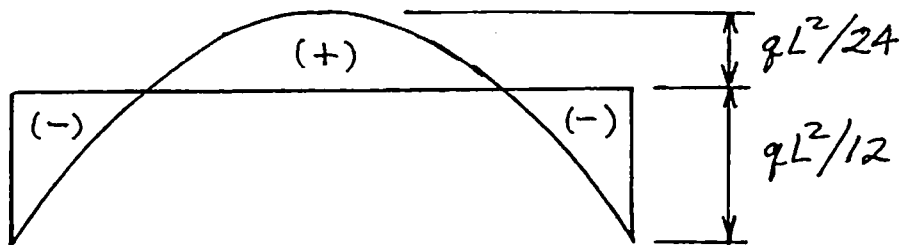
Vertical deflection at center node is the only nonzero d.o.f.

$$2 \frac{12EI}{a^3} v_2 = -qa, \quad v_2 = -\frac{qa^4}{24EI}$$

$$\text{or } v_2 = -\frac{qL^4}{384EI}$$



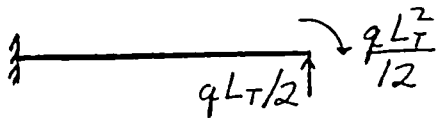
Element moments



Sum of the two (exact)

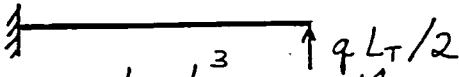
2.9-4

(a) Exact: $v = \frac{qL_T^4}{8EI}$, $M = \frac{qL_T^2}{2}$



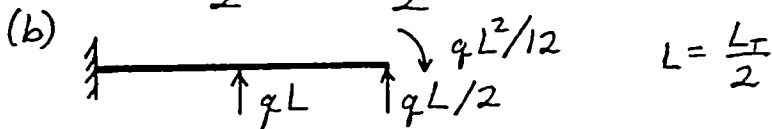
$$v = \frac{qL_T}{2} \frac{L_T^3}{3EI} - \frac{qL_T^2}{12} \frac{L_T}{2EI} = \frac{qL_T^4}{8EI} \quad \text{exact}$$

$$M = \frac{qL_T}{2} L_T - \frac{qL_T^2}{12} = 0.417 qL_T^2 \quad -16.7\%$$



$$v = \frac{qL_T}{2} \frac{L_T^3}{3EI} = \frac{qL_T^4}{6EI} \quad \text{---} \quad +33.3\%$$

$$M = \frac{qL_T}{2} L_T = \frac{qL_T^2}{2} \quad \text{---} \quad \text{exact}$$



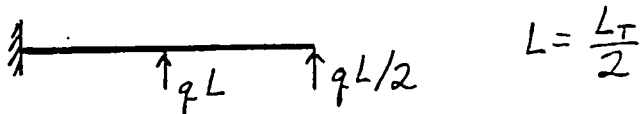
$$v = v_{\text{center force}} + v_{\text{end force}} + v_{\text{end moment}}$$

$$v = \left(qL \frac{L^3}{3EI} + qL \frac{L^2}{2EI} L \right) + \left(\frac{qL}{2} \frac{(2L)^3}{3EI} \right) - \left(\frac{qL^2}{12} \frac{(2L)^2}{2EI} \right) = \frac{qL^4}{EI} \left(\frac{5}{6} + \frac{4}{3} - \frac{1}{6} \right)$$

$$v = \frac{2qL^4}{EI} = \frac{qL_T^4}{8EI} \quad \text{---} \quad \text{exact}$$

$$M = qL(L) + \frac{qL}{2}(2L) - \frac{qL^2}{12}$$

$$M = \frac{23qL^2}{12} = \frac{23qL_T^2}{48} = 0.479 qL_T^2 \quad -4.2\%$$



Omit $v_{\text{end moment}}$ from foregoing

$$v = \frac{qL^4}{EI} \left(\frac{5}{6} + \frac{4}{3} \right) = \frac{13qL^4}{6EI} = 0.135 \frac{qL_T^4}{EI} \quad +8.3\%$$

$$M = qL(L) + \frac{qL}{2} 2L = 2qL^2 = \frac{qL_T^2}{2} \quad \text{exact}$$

2.9-5

$$(a) \begin{Bmatrix} v_2 \\ \theta_{22} \end{Bmatrix} = [K]^{-1} \{R\} = \frac{L}{6EI_2} \begin{bmatrix} 2L^2 & 3L \\ 3L & 6 \end{bmatrix} \begin{Bmatrix} F_2 \\ M_2 \end{Bmatrix} \quad \text{where} \quad \begin{aligned} F_2 &= P \left(\frac{3a^2}{L^2} - \frac{2a^3}{L^3} \right) \\ M_2 &= P \left(\frac{a^3}{L^2} - \frac{a^2}{L} \right) \end{aligned}$$

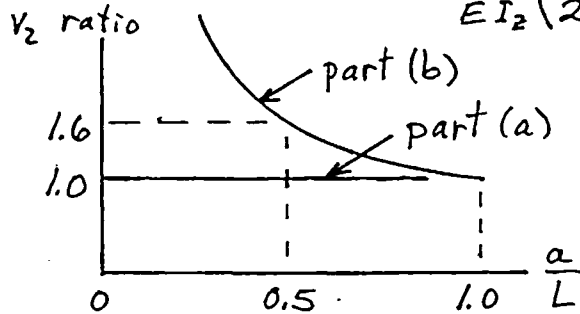
from which $v_2 = \frac{Pa^2}{EI_2} \left(\frac{L}{2} - \frac{a}{6} \right)$, $\theta_{22} = \frac{Pa^2}{2EI_2}$

$$(b) \begin{Bmatrix} v_2 \\ \theta_{22} \end{Bmatrix} = [K]^{-1} \{R\} = \frac{L}{6EI_2} \begin{bmatrix} 2L^2 & 3L \\ 3L & 6 \end{bmatrix} \begin{Bmatrix} Pa/L \\ 0 \end{Bmatrix} = \frac{PaL}{EI_2} \begin{Bmatrix} L/3 \\ 1/2 \end{Bmatrix}$$

$$(c) \text{ Beam theory: } v_2 = \frac{Pa^3}{3EI_2} + \frac{Pa^2}{2EI_2}(L-a) = \frac{Pa^2}{EI_2} \left(\frac{L}{2} - \frac{a}{6} \right)$$

Deflection ratios: in part (a), unity [part (a) exact]

$$\text{In part (b), ratio} = \frac{PaL^2/3EI_2}{\frac{Pa^2}{EI_2} \left(\frac{L}{2} - \frac{a}{6} \right)} = \frac{2}{\frac{a}{L} \left(3 - \frac{a}{L} \right)}$$



$$(d) \text{ Beam theory: } M_1 = Pa$$

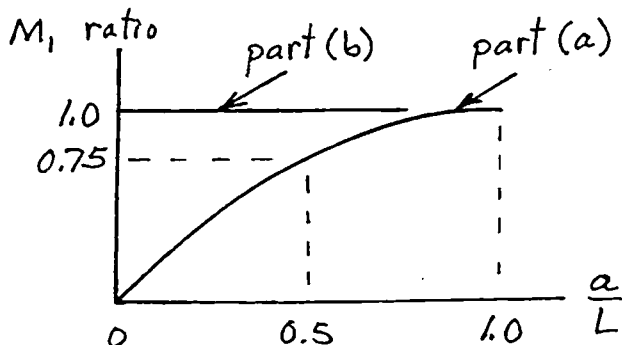
Moment ratio, part (a): from FEA,

$$M_1 = EI_2 \left[\left(\frac{6}{L^2} \right) \frac{Pa^2}{EI_2} \left(\frac{L}{2} - \frac{a}{6} \right) + \left(-\frac{2}{L} \right) \frac{Pa^2}{2EI_2} \right] = \frac{Pa^2}{L} \left(2 - \frac{a}{L} \right)$$

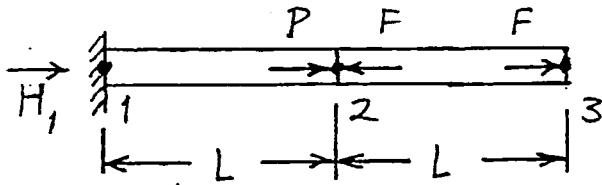
$$\text{moment ratio} = \frac{a}{L} \left(2 - \frac{a}{L} \right)$$

And in part (b): from FEA,

$$M_1 = EI_2 \left[\left(\frac{6}{L^2} \right) \frac{PaL^2}{3EI_2} + \left(-\frac{2}{L} \right) \frac{PaL}{2EI_2} \right] = Pa, \quad \text{moment ratio} = 1 \quad (\text{exact})$$



2.10-1



$$\sigma_0 = -E\alpha \frac{T_2 + T_3}{2} \quad (\text{right element only})$$

$$F = |A\sigma_0| = AE\alpha \frac{T_2 + T_3}{2}$$

In Eq. 2.10-3, set $u_1 = 0$ and $H_3 = 0$. Thus

$$\frac{AE}{L} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P - EA\alpha(T_2 + T_3)/2 \\ EA\alpha(T_2 + T_3)/2 \end{Bmatrix}$$

from which $u_2 = \frac{PL}{AE}$, $u_3 = \frac{PL}{AE} + \alpha L \frac{T_2 + T_3}{2}$

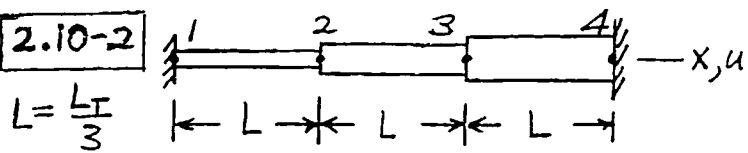
1st of Eqs. 2.10-3 then gives $H_1 = -P$

Stresses:

$$\sigma_{1-2} = E \frac{u_2}{L} + (\text{zero}) = \frac{P}{A} \quad \checkmark$$

$$\sigma_{2-3} = E \frac{u_3 - u_2}{L} + \sigma_0 = E\alpha \frac{T_2 + T_3}{2} + \left(-E\alpha \frac{T_2 + T_3}{2}\right) = 0 \quad \checkmark$$

2.10-2



Thermal loads at nodes 2 and 3 are

$$\begin{aligned}
 &\xrightarrow{\quad 2 \quad} \quad \quad \quad \xrightarrow{\quad 2 \quad} \\
 &\alpha E(1.1A_0)\Delta T \quad \alpha E(1.3A_0)\Delta T = 0.2\alpha EA_0\Delta T \\
 &\xrightarrow{\quad 3 \quad} \quad \quad \quad \xrightarrow{\quad 3 \quad} \\
 &\alpha E(1.3A_0)\Delta T \quad \alpha E(1.5A_0)\Delta T = 0.2\alpha EA_0\Delta T
 \end{aligned}$$

k is given by Eq. 2.2-7, with A being $1.1A_0$, $1.3A_0$, and $1.5A_0$ for the respective elements.

$$\begin{aligned}
 \tilde{K} &= \frac{A_0 E}{L} \left(\begin{bmatrix} 1.1 & -1.1 & 0 & 0 \\ -1.1 & 1.1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1.3 & -1.3 & 0 \\ 0 & -1.3 & 1.3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right. \\
 &\quad \left. + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & -1.5 \\ 0 & 0 & -1.5 & 1.5 \end{bmatrix} \right) = \frac{A_0 E}{L} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & 2.4 & -1.3 & \cdot \\ \cdot & -1.3 & 2.8 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}
 \end{aligned}$$

For the nonzero d.o.f. u_2 and u_3

$$\frac{A_0 E}{L} \begin{bmatrix} 2.4 & -1.3 \\ -1.3 & 2.8 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} -0.2 \\ -0.2 \end{Bmatrix} \alpha EA_0 \Delta T$$

$$\text{Solve: } \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \alpha L \Delta T \begin{Bmatrix} -0.163 \\ -0.147 \end{Bmatrix}$$

Initial stress is $-E\alpha\Delta T$ in each el.
each element:

$$\sigma_1 = E \frac{u_2}{L} + \sigma_0 = -1.163 E \alpha \Delta T$$

$$\sigma_2 = E \frac{u_3 - u_2}{L} + \sigma_0 = -0.984 E \alpha \Delta T$$

$$\sigma_3 = E \frac{u_4 - u_3}{L} + \sigma_0 = -0.853 E \alpha \Delta T$$

Nodal average stresses:

$$\text{Node 2, } \frac{\sigma_1 + \sigma_2}{2} = -1.074 E \alpha \Delta T$$

$$\text{Node 3, } \frac{\sigma_2 + \sigma_3}{2} = -0.919 E \alpha \Delta T$$

Exact solution: compute axial force P that negates free thermal expansion.

$$L_T \alpha \Delta T = \int_0^{L_T} \frac{P dx}{AE} = \frac{P}{E} \int_0^{L_T} \frac{dx}{A_0(1+0.6\frac{x}{L_T})}$$

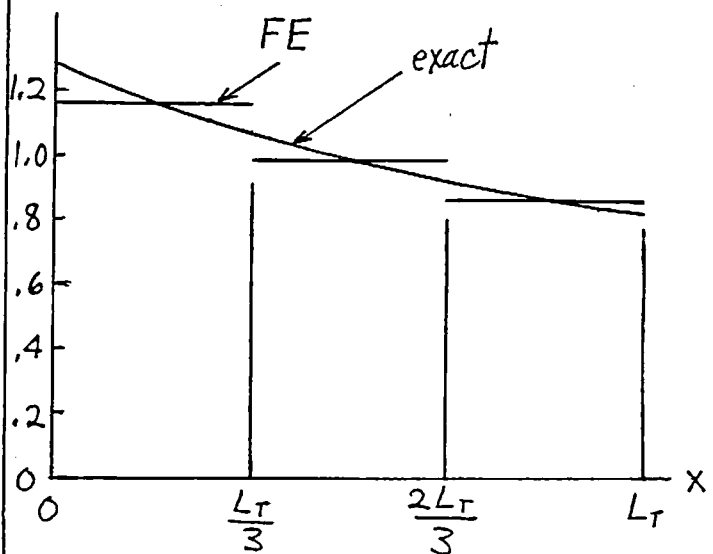
$$L_T \alpha \Delta T = \frac{PL_T}{EA_0(0.6)} \ln \left(1 + 0.6 \frac{x}{L_T} \right) \Big|_0^{L_T}$$

$$L_T \alpha \Delta T = \frac{PL_T}{0.6 EA_0} \ln 1.6 = 0.783 \frac{PL_T}{EA_0}$$

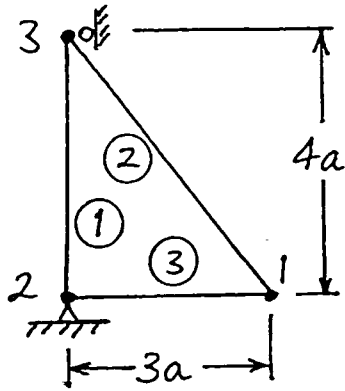
$$P = 1.277 \alpha EA_0 \Delta T \quad (\text{compressive})$$

location	A	$\sigma = P/A$
node 1	A_0	$-1.277 \alpha E \Delta T$
el. 1	$1.1A_0$	$-1.161 \alpha E \Delta T$
node 2	$1.2A_0$	$-1.064 \alpha E \Delta T$
el. 2	$1.3A_0$	$-0.982 \alpha E \Delta T$
node 3	$1.4A_0$	$-0.912 \alpha E \Delta T$
el. 3	$1.5A_0$	$-0.851 \alpha E \Delta T$
node 4	$1.6A_0$	$-0.798 \alpha E \Delta T$

$\alpha E \Delta T$



2.10-3



Heat bar 2 only. In that bar,

$$\sigma_0 = -E\alpha T, \quad F = |A\sigma_0| = EA\alpha T$$

Eq. 2.7-8 becomes

$$\begin{bmatrix} k_3 + 0.36k_2 & -0.48k_2 & 0.48k_2 \\ -0.48k_2 & 0.64k_2 & -0.64k_2 \\ 0.48k_2 & -0.64k_2 & k_1 + 0.64k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0.6F \\ -0.8F \\ 0.8F \end{Bmatrix}$$

$$\text{where } k_2 = \frac{AE}{5a}$$

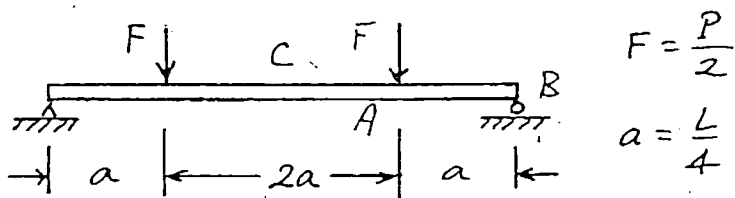
$$\text{Solution is } \begin{Bmatrix} u_1 \\ v_1 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -6.25a\alpha T \\ 0 \end{Bmatrix}$$

Bar stresses: $\sigma_1 = 0$, $\sigma_3 = 0$, and

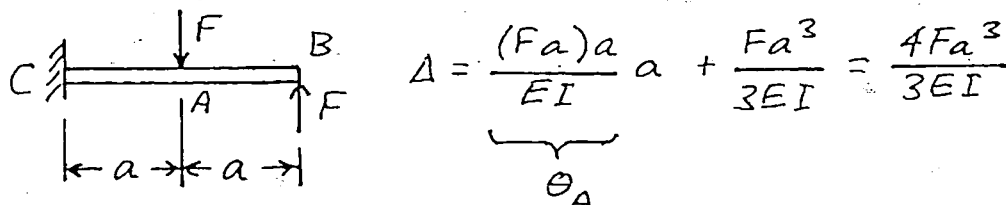
$$\sigma_2 = \frac{E}{5a} (0.8)(6.25a\alpha T) + (-E\alpha T) = 0$$

2.11-1

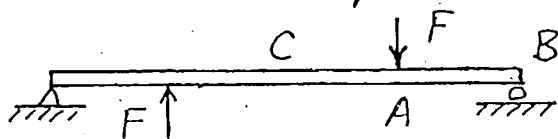
First case (symmetric loads)



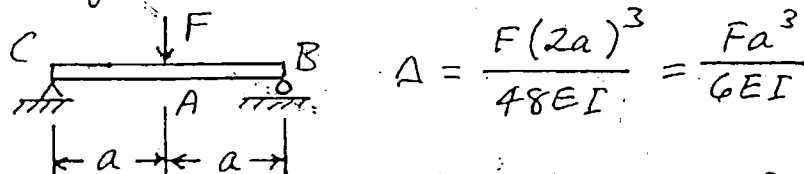
Consider right half to get deflection of A relative to B



Second case (antisymmetric loads)



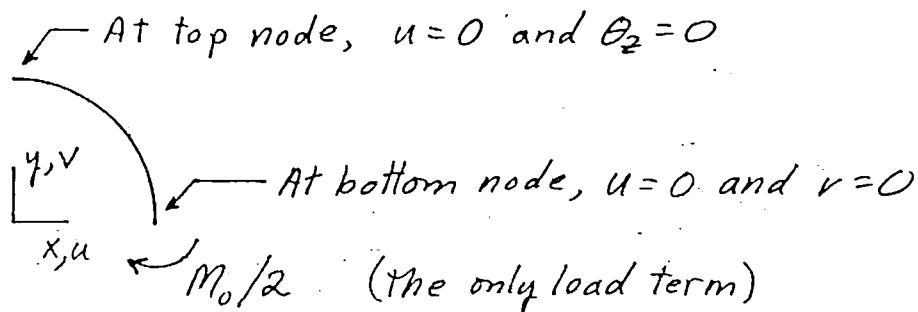
Again consider right half



$$\text{Sum} = \frac{Fa^3}{EI} \left(\frac{4}{3} + \frac{1}{6} \right) = \frac{3Fa^3}{2EI} = \frac{3}{2EI} \frac{P}{2} \left(\frac{L}{4} \right)^3 = \frac{3PL^3}{256EI}$$

Handbook formula: $\frac{P(3L/4)^2(L/4)^2}{3EIL} = \frac{3PL^3}{256EI}$

2.11-2



Also prevent out-of-plane motion: at (say) top node, set $w = \theta_x = \theta_y = 0$.

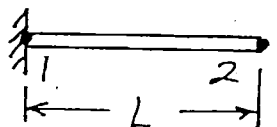
(a) Symmetric case:

Restrain at $x=0$: u, θ_y, θ_z Zero loads at $x=0$: y -direction transverse shear force
 z -direction transverse shear force
torque

(b) Antisymmetric case:

Restrain at $x=0$: v, w, θ_x Zero loads at $x=0$: x -direction force
moment M_y
moment M_z

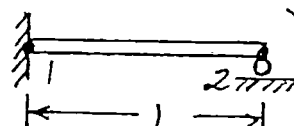
2.10-4 With no deformation, stress is $-E\alpha T$ on top surface, $+E\alpha T$ on bottom surface. The associated bending moment is $M_z = \frac{\sigma I}{c} = E\alpha T \frac{t(2c)^3}{12c} = \frac{2}{3} E\alpha T t c^2$

(a)  M_z for which $v_2 = -\frac{M_z L^2}{2EI}$, $\theta_2 = -\frac{M_z L}{EI}$

With $v_1 = \theta_1 = 0$, Eq. 2.9-4 gives

$$M = EI \left[\left(\frac{6}{L^2} - \frac{12x}{L^3} \right) \left(-\frac{M_z L^2}{2EI} \right) + \left(-\frac{2}{L} + \frac{6x}{L^2} \right) \left(-\frac{M_z L}{EI} \right) \right] = -M_z$$

Net M , from nodal d.o.f. + temperature change, is zero, for all x
so zero stress is predicted.

(b)  M_z for which $\frac{4EI}{L} \theta_2 = -M_z$, $\theta_2 = -\frac{M_z L}{4EI}$

With θ_2 the only nonzero d.o.f., Eq. 2.9-4 gives

$$M = EI \left(-\frac{2}{L} + \frac{6x}{L^2} \right) \left(-\frac{M_z L}{4EI} \right) = \frac{M_z}{2} \left(1 - \frac{3x}{L} \right)$$

Net M , including M from temperature change, is

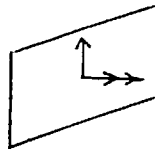
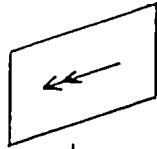
$$\frac{M_z}{2} \left(1 - \frac{3x}{L} \right) + M_z = \frac{3M_z}{2} \left(1 - \frac{x}{L} \right) = E\alpha T t c^2 \left(1 - \frac{x}{L} \right)$$

On the top surface,

$$\sigma = -\frac{Mc}{I} = -\frac{c}{t(2c)^3/12} E\alpha T t c^2 \left(1 - \frac{x}{L} \right)$$

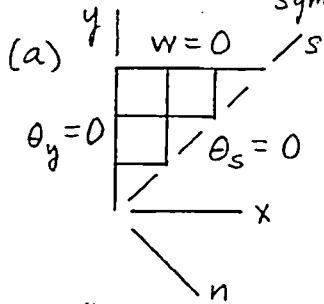
2.11-4

D.o.f. restrained:



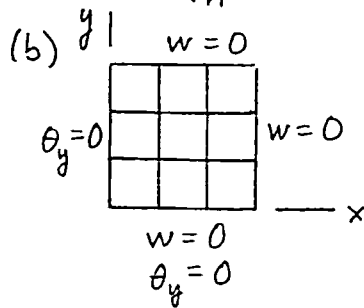
symmetry

antisymmetry



Symmetry w.r.t.
 $x=0$ and $n=0$
planes

Apply loads $P/2$ to
nodes on symm. planes.



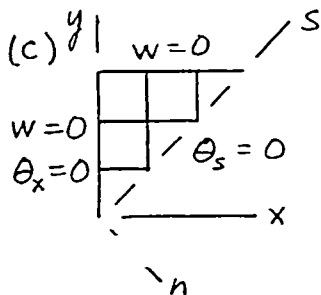
Symmetry w.r.t.

$x=0$ plane

Antisymmetry w.r.t.
 $y=0$ plane

Apply loads $P/2$ to
nodes on $x=0$.

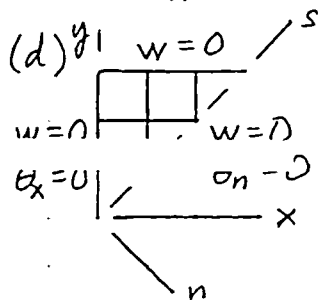
Could go further: cut the quadrant in
half by an x -parallel line; impose
symmetry ($\theta_x=0$) along this cut.



Symmetry w.r.t.

$n=0$ plane

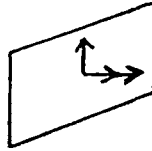
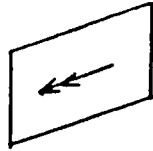
Antisymmetry
w.r.t. $x=0$ plane



Antisymmetry w.r.t.

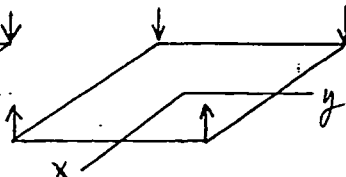
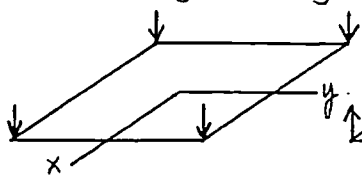
$x=0$ and $n=0$
planes

D.o.f. restrained:



symmetry . antisymmetry

Analyze the following four cases, with each load = $P/4$. Only one quadrant (e.g. the first) need be treated. Deflections in quadrants 2, 3, 4 can be obtained from deflections in quadrant 1 by use of the symmetry and antisymmetry conditions noted.



$$\theta_y = 0 \text{ on } x=0 \text{ plane}$$

$$\left. \begin{matrix} w=0 \\ \theta_x=0 \end{matrix} \right\} \text{ on } x=0 \text{ plane}$$

$$\theta_x = 0 \text{ on } y=0 \text{ plane}$$

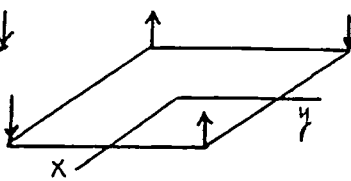
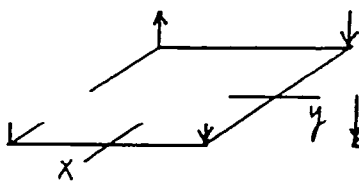
$$\theta_x = 0 \text{ on } y=0 \text{ plane}$$

$$w(x) = w(-x)$$

$$w(x) = -w(-x)$$

$$w(y) = w(-y)$$

$$w(y) = w(-y)$$



$$\theta_y = 0 \text{ on } x=0 \text{ plane}$$

$$\left. \begin{matrix} w=0 \\ \theta_x=0 \end{matrix} \right\} \text{ on } x=0 \text{ plane}$$

$$w=0 \text{ on } y=0 \text{ plane}$$

$$\left. \begin{matrix} w=0 \\ \theta_y=0 \end{matrix} \right\} \text{ on } y=0 \text{ plane}$$

$$\theta_y = 0$$

$$w(x) = w(-x)$$

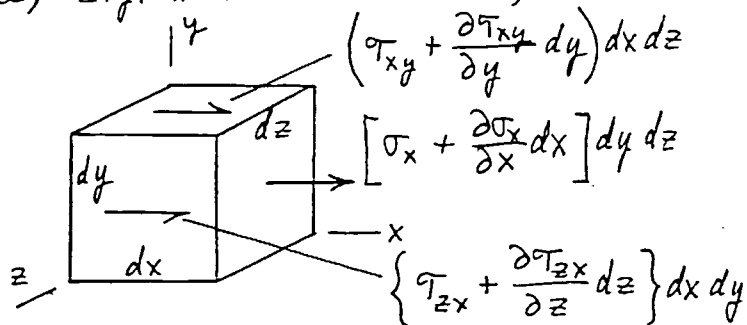
$$w(x) = -w(-x)$$

$$w(y) = -w(-y)$$

$$w(y) = -w(-y)$$

3.1-1

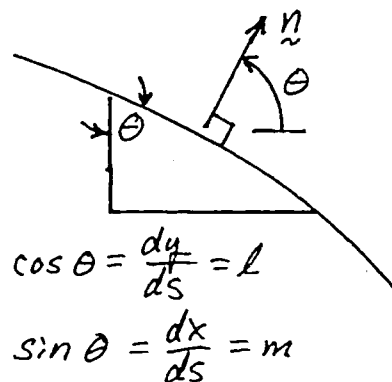
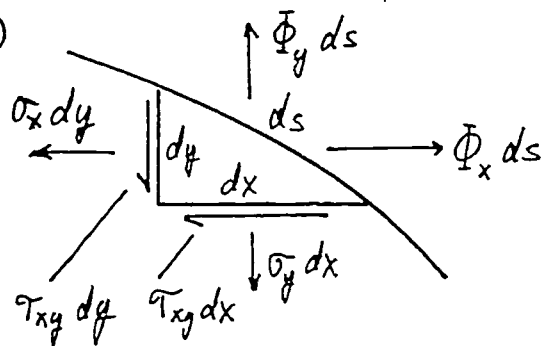
(a) E.g. in the x direction,



$$\sum F_{x \text{ acts}} = 0 = -\sigma_x dy dz - \tau_{xy} dx dz - \tau_{xz} dx dy + \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx \right) dy dz + \left\{ \tau_{xz} + \frac{\partial \tau_{xz}}{\partial z} dz \right\} dx dy + F_x dx dy dz \quad \text{yields}$$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x = 0$$

(b)



Thickness immaterial; can take as unity.

$$\sum F_x = 0 = -\sigma_x dy - \tau_{xy} dx + \Phi_x ds$$

$$\sum F_y = 0 = -\sigma_y dx - \tau_{xy} dy + \Phi_y ds$$

$$\Phi_x = \sigma_x \frac{dy}{ds} + \tau_{xy} \frac{dx}{ds}$$

$$\Phi_y = \tau_{xy} \frac{dy}{ds} + \sigma_y \frac{dx}{ds}$$

$$\Phi_x = l \sigma_x + m \tau_{xy}$$

$$\Phi_y = l \tau_{xy} + m \sigma_y$$

3.1-2

(a)

$$\sigma_x = -6a, x^2, \sigma_y = 12a, y^2, \tau_{xy} = 12a, xy^2$$

1 4 b

$$\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} b$$

On 1-2: $l=1, m=0, x=b$

$$\Phi_x = \sigma_x = -6a, b^2$$

$$\Phi_y = \tau_{xy} = 12a, y^2$$

On 2-3: $l=0, m=1, y=b$

$$\Phi_x = \tau_{xy} = 12a, b^2, \Phi_y = \sigma_y = 12a, x^2$$

On 3-4: $l=-1, m=0, x=0$

$$\Phi_x = 0, \Phi_y = -\tau_{xy} = -12a, y^2$$

On 4-1: $l=0, m=-1, y=0$

$$\Phi_x = -\tau_{xy} = 0, \Phi_y = -\sigma_y = -12a, x^2$$

(b) Check equilibrium.

$$x\text{-dir. } \sigma_{x,x} + \tau_{xy,y} = -12a, x + 24a, y \neq 0$$

$$y\text{-dir. } \tau_{xy,x} + \sigma_{y,y} = 0 + 0 = 0 \quad \checkmark$$

Equil. not satisfied; field not possible.

3.1-3

$$\sigma_x = 3a, x^2 y, \sigma_y = a, y^3, \tau_{xy} = -3a, xy^2$$

Equilibrium:

$$\sigma_{x,x} + \tau_{xy,y} = 6a, xy - 6a, xy = 0 \quad \checkmark$$

$$\tau_{xy,x} + \sigma_{y,y} = -3a, y^2 + 3a, y^2 = 0 \quad \checkmark$$

Check compatibility:

$$\epsilon_x = \frac{1}{E} (3a, x^2 y - \nu a, y^3)$$

$$\epsilon_y = \frac{1}{E} (a, y^3 - 3\nu a, x^2 y)$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E} (-3a, xy^2)$$

$$\epsilon_{x,yy} + \epsilon_{y,xx} \neq \gamma_{xy,xy}$$

$$\frac{1}{E} (-6\nu a, y - 6\nu a, y) \neq \frac{2(1+\nu)}{E} (-6a, y)$$

$\nu \neq 1+\nu$ Not possible.

3.1-5

$$\epsilon_x = u_{,x} = a_2 + 2a_4 x + a_5 y$$

$$\epsilon_y = v_{,y} = a_9 + a_{11} x + 2a_{12} y$$

$$\gamma_{xy} = u_{,y} + v_{,x} = a_3 + a_5 x + 2a_6 y + a_8 + 2a_{10} x + a_{11} y$$

$$\text{Let } A = \frac{E}{1-\nu^2}. \text{ Then}$$

$$\sigma_x = A(\epsilon_x + \nu \epsilon_y), \quad \sigma_y = A(\epsilon_y + \nu \epsilon_x), \quad \tau_{xy} = G \gamma_{xy}$$

Equilibrium eqs. become

$$\sigma_{x,x} + \tau_{xy,y} = A(2a_4 + \nu a_{11}) + G(2a_6 + a_{11})$$

$$\tau_{xy,x} + \sigma_{y,y} = G(a_5 + 2a_{10}) + A(2a_{12} + \nu a_5)$$

When are these 2 eqs. satisfied?

With $G = (1-\nu)A/2$,

$$2a_4 + \nu a_{11} + \frac{1-\nu}{2}(2a_6 + a_{11}) = 0$$

$$\frac{1-\nu}{2}(a_5 + 2a_{10}) + 2a_{12} + \nu a_5 = 0$$

$$\text{or } 2a_4 + (1-\nu)a_6 + \frac{1+\nu}{2}a_{11} = 0$$

$$2a_{12} + (1-\nu)a_{10} + \frac{1+\nu}{2}a_5 = 0$$

3.1-4

$$(a) \Delta V = V_{\text{final}} - V_{\text{original}}$$

$$\Delta V = [(1+\epsilon_x)dx(1+\epsilon_y)dy(1+\epsilon_z)dz] - [dx dy dz]$$

$$\Delta V = (\epsilon_x + \epsilon_y + \epsilon_z)dx dy dz + \text{higher order terms}$$

$$\Delta V = (\epsilon_x + \epsilon_y + \epsilon_z)dx dy dz \quad \text{for small strains}$$

$$\frac{\Delta V}{V} = \frac{\Delta V}{dx dy dz} = \epsilon_x + \epsilon_y + \epsilon_z$$

$$(b) \text{ Use 3D form of Eqs. 3.1-4, with } \sigma_x = \sigma_y = \sigma_z = -p:$$

$$\epsilon_x = \frac{-p}{E}(1-2\nu) \quad \epsilon_y = \epsilon_x \quad \epsilon_z = \epsilon_x$$

$$\text{Hence } \frac{\Delta V}{V} = -\frac{3p}{E}(1-2\nu) \quad \text{and} \quad \frac{\Delta V/V}{p} = -\frac{3(1-2\nu)}{E}$$

$$(c) \text{ As } \nu \rightarrow 0.5, \quad \frac{\Delta V}{V} \rightarrow 0$$

3.1-4

(a) $\Delta V = V_{\text{final}} - V_{\text{original}}$

$$\Delta V = [(1 + \epsilon_x)dx(1 + \epsilon_y)dy(1 + \epsilon_z)dz] - [dx dy dz]$$

$$\Delta V = (\epsilon_x + \epsilon_y + \epsilon_z)dx dy dz + \text{higher order terms}$$

$$\Delta V = (\epsilon_x + \epsilon_y + \epsilon_z)dx dy dz \quad \text{for small strains}$$

$$\frac{\Delta V}{V} = \frac{\Delta V}{dx dy dz} = \epsilon_x + \epsilon_y + \epsilon_z$$

(b) Use 3D form of Eqs. 3.1-4, with $\sigma_x = \sigma_y = \sigma_z = -p$:

$$\epsilon_x = \frac{-p}{E}(1 - 2\nu) \quad \epsilon_y = \epsilon_x \quad \epsilon_z = \epsilon_x$$

$$\text{Hence } \frac{\Delta V}{V} = -\frac{3p}{E}(1 - 2\nu) \quad \text{and} \quad \frac{\Delta V/V}{p} = -\frac{3(1 - 2\nu)}{E}$$

(c) As $\nu \rightarrow 0.5$, $\frac{\Delta V}{V} \rightarrow 0$

3.2-1

$$N_1 = \frac{(2-x)(3-x)}{(2)(3)} = \frac{1}{6}(6 - 5x + x^2)$$

$$N_2 = \frac{-x(3-x)}{(-2)(1)} = \frac{1}{2}(3x - x^2)$$

$$N_3 = \frac{-x(2-x)}{(-3)(-1)} = \frac{1}{3}(-2x + x^2)$$

(a) $\sum_1^3 N_i = \frac{1}{6}(6 - 5x + x^2 + 9x - 3x^2 - 4x + 2x^2) = 1$

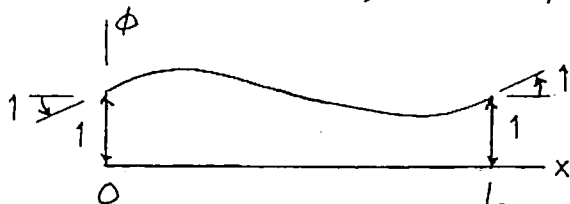
(b) $N_{1,x} = \frac{1}{6}(-5 + 2x) \quad \sum_1^3 N_{i,x} = \frac{1}{6}(-5 + 2x$

$$N_{2,x} = \frac{1}{2}(3 - 2x) \quad + 9 - 6x - 4 + 4x)$$

$$N_{3,x} = \frac{1}{3}(-2 + 2x) \quad = 0$$

3.2-2

If all 4 $\phi_i = 1$, the shape is



Clearly $\phi \neq 1$ throughout ($\sum N_i = 1$ if

$\sum N_i \phi_i = \phi$ gives $1 = 1$ for $\phi = \phi_i = 1$).

Also, not all N_i have the same units.

3.1-5

$$\epsilon_x = u_{,x} = a_2 + 2a_4x + a_5y$$

$$\epsilon_y = v_{,y} = a_9 + a_{11}x + 2a_{12}y$$

$$\gamma_{xy} = u_{,y} + v_{,x} = a_3 + a_5x + 2a_6y + a_8 + 2a_{10}x + a_{11}y$$

$$\text{Let } A = \frac{E}{1-\nu^2}. \text{ Then}$$

$$\sigma_x = A(\epsilon_x + \nu\epsilon_y), \sigma_y = A(\epsilon_y + \nu\epsilon_x), \tau_{xy} = G\gamma_{xy}$$

Equilibrium eqs. become

$$\sigma_{x,x} + \tau_{xy,y} = A(2a_4 + \nu a_{11}) + G(2a_6 + a_{11})$$

$$\tau_{xy,x} + \sigma_{y,y} = G(a_5 + 2a_{10}) + A(2a_{12} + \nu a_5)$$

When are these 2 eqs. satisfied?

With $G = (1-\nu)A/2$,

$$2a_4 + \nu a_{11} + \frac{1-\nu}{2}(2a_6 + a_{11}) = 0$$

$$\frac{1-\nu}{2}(a_5 + 2a_{10}) + 2a_{12} + \nu a_5 = 0$$

$$\text{or } 2a_4 + (1-\nu)a_6 + \frac{1+\nu}{2}a_{11} = 0$$

$$2a_{12} + (1-\nu)a_{10} + \frac{1+\nu}{2}a_5 = 0$$

3.2-3

(a) $\phi = \sum N_i \phi_i$; use Eqs. 3.2-7

$$\phi = \frac{(3-x)(5-x)(8-x)}{(2)(4)(7)} 2 + \frac{(1-x)(5-x)(8-x)}{(-2)(2)(5)} 2$$

$$+ \frac{(1-x)(3-x)(8-x)}{(-4)(-2)(3)} 2 + \frac{(1-x)(3-x)(5-x)}{(-7)(-5)(-3)} 5$$

$$(b) \phi_0 = \frac{3 \cdot 5 \cdot 8}{56} 2 + \frac{1 \cdot 5 \cdot 8}{-20} 2 + \frac{1 \cdot 3 \cdot 8}{24} 2 + \frac{1 \cdot 3 \cdot 5}{-105} 5$$

$$= \frac{120}{28} - \frac{40}{10} + \frac{24}{12} - \frac{15}{21} = 1.57$$

Similarly

$$\phi_2 = \frac{1 \cdot 3 \cdot 6}{28} - \frac{(-1) \cdot 3 \cdot 6}{10} + \frac{(-1) \cdot 1 \cdot 6}{12} - \frac{(-1) \cdot 1 \cdot 3}{21} = 2.086$$

$$\phi_4 = \frac{(-1) \cdot 1 \cdot 4}{28} - \frac{(-3) \cdot 1 \cdot 4}{10} + \frac{(-3) \cdot (-1) \cdot 4}{12} - \frac{(-3) \cdot (-1) \cdot 1}{21} = 1.914$$

$$\phi_7 = \frac{(-4) \cdot (-2) \cdot 1}{28} - \frac{(-6) \cdot (-2) \cdot 1}{10} + \frac{(-6) \cdot (-4) \cdot 1}{12} - \frac{(-6) \cdot (-4) \cdot (-2)}{21} = 3.371$$

3.2-4

First two of Eqs. 3.2-8 yield $a_1 = \phi_1$, $a_2 = \phi_{,x1}$

The second two equations become

$$\begin{bmatrix} L^2 & L^3 \\ 2L & 3L^2 \end{bmatrix} \begin{Bmatrix} a_3 \\ a_4 \end{Bmatrix} = \begin{Bmatrix} \phi_2 - \phi_1 - L\phi_{,x1} \\ \phi_{,x2} - \phi_{,x1} \end{Bmatrix},$$

$$\begin{Bmatrix} a_3 \\ a_4 \end{Bmatrix} = \frac{1}{L^4} \begin{bmatrix} 3L^2 & -L^3 \\ -2L & L^2 \end{bmatrix} \begin{Bmatrix} \phi_2 - \phi_1 - L\phi_{,x1} \\ \phi_{,x2} - \phi_{,x1} \end{Bmatrix} = \begin{Bmatrix} (-3\phi_1 - 2L\phi_{,x1} + 3\phi_2 - L\phi_{,x2})/L^2 \\ (2\phi_1 + L\phi_{,x1} - 2\phi_2 + L\phi_{,x2})/L^3 \end{Bmatrix}$$

$$[\tilde{A}]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/L^2 & -2/L & 3/L^2 & -1/L \\ 2/L^3 & 1/L^2 & -2/L^3 & 1/L^2 \end{bmatrix}$$

with $[\tilde{X}] = [1 \ x \ x^2 \ x^3]$,

$$[\tilde{X}][\tilde{A}]^{-1} = \begin{bmatrix} 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} & x - \frac{2x^2}{L} + \frac{x^3}{L^2} & \frac{3x^2}{L^2} - \frac{2x^3}{L^3} & -\frac{x^2}{L} + \frac{x^3}{L^2} \end{bmatrix}$$

$N_1 \qquad N_2 \qquad N_3 \qquad N_4$

3.2-5

$$\phi = [X] \{a\} = [1 \ x \ x^2] \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\phi_{,x} = [0 \ 1 \ 2x] \{a\}$$

Evaluate given conditions (let $\theta_1 = \phi_{,x1}$)

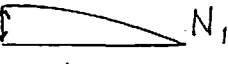
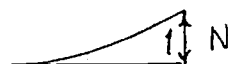
$$\begin{Bmatrix} \phi_1 \\ \theta_1 \\ \phi_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & L & L^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \quad \begin{array}{l} a_1 = \phi_1, a_2 = \theta_1 \\ 3^{rd} \text{ eq. is then} \\ \phi_2 - \phi_1 - L\theta_1 = L^2 a_3 \end{array}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{L^2} & -\frac{1}{L} & \frac{1}{L^2} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \theta_1 \\ \phi_2 \end{Bmatrix} \quad \therefore a_3 = \frac{\phi_2 - \phi_1}{L^2} - \frac{\theta_1}{L}$$

$$\phi = [X] [A]^{-1} \{d\} = [N] \{d\} \text{ where}$$

$$[N] = \begin{bmatrix} 1 - \frac{x^2}{L^2} & x(1 - \frac{x}{L}) & \frac{x^2}{L^2} \end{bmatrix} \text{ Also}$$

$$[N_{,x}] = \begin{bmatrix} -\frac{2x}{L^2} & 1 - \frac{2x}{L} & \frac{2x}{L^2} \end{bmatrix}$$

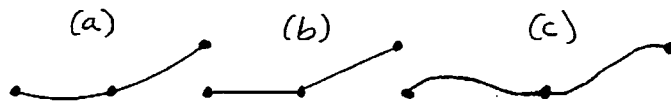
i	At x=0		At x=L		
	N_i	$N_{i,x}$	N_i	$N_{i,x}$	
1	1	0	0	-2/L	
2	0	1	0	-1	
3	0	0	1	2/L	

3.2-6

Lagrange: 3 pts. define parabola.

Piecewise C^0 : two straight lines.

Piecewise C^1 : two cubics.



3.3-1

$$(a) \quad u = \begin{bmatrix} 1 & x \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}, \quad [B_a] = \frac{\partial}{\partial x} \begin{bmatrix} 1 & x \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$[k_a] = \int_0^L [B_a]^T [B_a] AE dx = AEL \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Let $x_1 = 0, x_2 = L$ in Eq. 3.2-4. Then $[A] = \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix}, [A]^{-1} = \begin{bmatrix} 1 & 0 \\ -1/L & 1/L \end{bmatrix}$

$$[k] = [A]^{-T} [k_a] [A]^{-1} = \begin{bmatrix} 1 & -1/L \\ 0 & 1/L \end{bmatrix} \left(AEL \begin{bmatrix} 0 & 0 \\ -1/L & 1/L \end{bmatrix} \right) = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \checkmark$$

$$(b) \quad v = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}, \quad \frac{d^2 v}{dx^2} = \begin{bmatrix} 0 & 0 & 2 & 6x \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} = [B_a] \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}$$

$$[k_a] = \int_0^L [B_a]^T [B_a] EI dx = EI \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4L & 6L^2 \\ 0 & 0 & 6L^2 & 12L^3 \end{bmatrix}$$

$[A]^{-1}$ is calculated in Problem 3.2-4

$$[k] = [A]^{-1} [k_a] [A]^{-1} = \begin{bmatrix} 1 & 0 & -3/L^2 & 2/L^3 \\ 0 & 1 & -2/L & 1/L^2 \\ 0 & 0 & 3/L^2 & -2/L^3 \\ 0 & 0 & -1/L & 1/L^2 \end{bmatrix} \left(EI \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 \\ 6 & 0 & -6 & 6L \end{bmatrix} \right)$$

$$[k] = EI \begin{bmatrix} 12/L^3 & 6/L^2 & -12/L^3 & 6/L^2 \\ 6/L^2 & 4/L & -6/L^2 & 2/L \\ -12/L^3 & -6/L^2 & 12/L^3 & -6/L^2 \\ 6/L^2 & 2/L & -6/L^2 & 4/L \end{bmatrix} \checkmark$$

3.3-2

(a)

Follow the method of Prob. 3.3-1. That is, use the "a-basis", $u = [1 \ x \ x^2 \ x^3] \{a\}$,

$$[k_a] = \int_0^L [B_a]^T [B_a] AE dx, \text{ where } [B_a] =$$

$$[0 \ 1 \ 2x \ 3x^2], [k] = [A]^{-T} [k_a] [A]^{-1},$$

and $[A]^{-1}$ is computed in Prob. 3.2-4.

$$AE \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & L & L^2 & L^3 \\ 0 & L^2 & \frac{4}{3}L^3 & \frac{3}{2}L^4 \\ 0 & L^3 & \frac{3}{2}L^4 & \frac{9}{5}L^5 \end{bmatrix}, AE \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -L & -\frac{L^2}{6} & L & \frac{L^2}{6} \\ -9L^2 & -2L^3 & 9L^2 & 3L^3 \end{bmatrix}$$

$[k_a] \qquad [k_a][A]^{-1}$

$$\begin{bmatrix} 1 & 0 & -3/L^2 & 2/L^3 \\ 0 & 1 & -2/L & 1/L^2 \\ 0 & 0 & 3/L^2 & -2/L^3 \\ 0 & 0 & -1/L & 1/L^2 \end{bmatrix}, \frac{AE}{30L} \begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix}$$

$[A]^{-T} \qquad [k]$

(b) Must not enforce interelement continuity of ϵ_x at these locations. Where two elements meet, can condense the ϵ_x d.o.f. in one of them before assembly.

3.3-3 Eqs. 3.2-7: $x_1 = 0, x_2 = \frac{L}{2}, x_3 = L$

$$N_1 = \frac{(\frac{L}{2} - x)(L - x)}{\frac{L}{2}L} = \frac{1}{L^2}(L^2 - 3Lx + 2x^2)$$

$$N_2 = \frac{(-x)(L - x)}{-\frac{L}{2}(\frac{L}{2})} = \frac{4}{L^2}(Lx - x^2)$$

$$N_3 = \frac{-x(\frac{L}{2} - x)}{-L(-\frac{L}{2})} = \frac{1}{L^2}(2x^2 - Lx)$$

$$[B] = \frac{d}{dx}[N] = \frac{1}{L^2} \begin{bmatrix} -3L + 4x & 4(L - 2x) & 4x - L \end{bmatrix}$$

$$[k] = \int_0^L [B]^T [B] AE dx$$

$$[k] = \frac{AE}{L^4} \int_0^L \begin{bmatrix} 9L - 24Lx + 16x^2 & -12L^2 + 40Lx - 32x^2 & 3L^2 - 16Lx + 16x^2 \\ \text{symmetric} & 16(L^2 - 4Lx + 4x^2) & 4(-L^2 + 6Lx - 8x^2) \\ & & L^2 - 8Lx + 16x^2 \end{bmatrix} dx$$

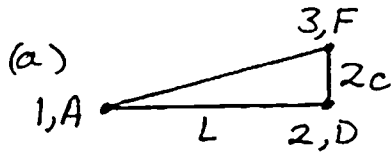
$$[k] = \frac{AE}{L^4} \begin{bmatrix} 9L^2x - 12Lx^2 + \frac{16}{3}x^3 & -12L^2x + 20Lx^2 - \frac{32}{3}x^3 & 3L^2x - 8Lx^2 + \frac{16}{3}x^3 \\ \text{symmetric} & 16(L^2x - 2Lx^2 + \frac{4}{3}x^3) & 4(-L^2x + 3Lx^2 - \frac{8}{3}x^3) \\ & & L^2x - 4Lx^2 + \frac{16}{3}x^3 \end{bmatrix}_0^L$$

$$[k] = \frac{AE}{L^4} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

3.4-1 Beam theory: $I = \frac{t}{12}(2c)^3 = \frac{2tc^3}{3}$

$$V_D = V_F = -\frac{ML^2}{2EI} = -\frac{3ML^2}{4Etc^3}$$

$$u_F = -u_D = c \frac{ML}{EI} = \frac{3ML}{2Etc^2}$$



Obtain $[B]$ from Eq. 3.4-10

$$\{\underline{\epsilon}\} = [B]\{\underline{d}\} = \begin{bmatrix} -1/L & 0 & 1/L & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/2c & 0 & 1/2c \\ 0 & -1/L & -1/2c & 1/L & 1/2c & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_D \\ v_D \\ u_F \\ v_F \end{Bmatrix}$$

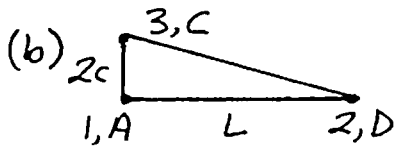
$$\epsilon_x = \frac{u_D}{L} = -\frac{3M}{2Etc^2}, \quad \epsilon_y = \frac{v_F - v_D}{2c} = 0,$$

$$\gamma_{xy} = -\frac{u_D}{2c} + \frac{v_D}{L} + \frac{u_F}{2c} = \frac{u_F}{c} + \frac{v_D}{L} = \frac{3ML}{4Etc^3}$$

With $\nu = 0$, $\sigma_x = E\epsilon_x = \frac{3M}{2tc^2}$, $\sigma_y = E\epsilon_y = 0$, $\tau_{xy} = \frac{E}{2}\gamma_{xy} = \frac{3ML}{4tc^3}$

Beam theory, top & bottom: $|\sigma_x| = \frac{Mc}{8tc^3/12} = \frac{3M}{2tc^2}$

With σ_x & τ_{xy} from FEA, $\frac{\tau_{xy}}{\sigma_x} = \frac{L}{2c} \rightarrow \infty$ as $\frac{L}{c} \rightarrow \infty$



Obtain $[B]$ from Eq. 3.4-10

$$\{\underline{\epsilon}\} = [B]\{\underline{d}\} = \begin{bmatrix} -1/L & 0 & 1/L & 0 & 0 & 0 \\ 0 & -1/2c & 0 & 0 & 0 & 1/2c \\ -1/2c & -1/L & 0 & 1/L & 1/2c & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_D \\ v_D \\ 0 \\ 0 \end{Bmatrix}$$

$$\epsilon_x = \frac{v_D}{L} = -\frac{3M}{2Etc^2}, \quad \epsilon_y = 0, \quad \gamma_{xy} = \frac{v_D}{L} = -\frac{3ML}{4Etc^3}$$

With $\nu = 0$, $\sigma_x = E\epsilon_x = -\frac{3M}{2tc^2}$, $\sigma_y = E\epsilon_y = 0$

$$\tau_{xy} = \frac{E}{2}\gamma_{xy} = -\frac{3ML}{8tc^3}$$

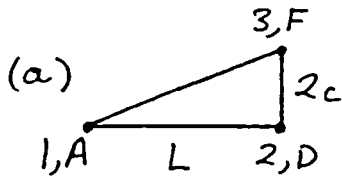
Beam theory: as in part (a)

With σ_x & τ_{xy} from FEA, $\frac{\tau_{xy}}{\sigma_x} = \frac{L}{4c} \rightarrow \infty$ as $\frac{L}{c} \rightarrow \infty$

3.4-2 Beam theory: $I = \frac{t}{12} (2c)^3 = \frac{2tc^3}{3}$

$$v_D = v_F = \frac{PL^3}{3EI} = \frac{PL^3}{2Etc^3}$$

$$u_F = -u_D = -c \frac{PL^2}{2EI} = -\frac{3PL^2}{4Etc^2}$$



Obtain $[B]$ from Eq. 3.4-10

$$\{\epsilon\} = [B]\{d\} = \begin{bmatrix} -1/L & 0 & 1/L & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/2c & 0 & 1/2c \\ 0 & -1/L & -1/2c & 1/L & 1/2c & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_D \\ v_D \\ u_F \\ v_F \end{Bmatrix}$$

$$\epsilon_x = \frac{u_D}{L} = \frac{3PL}{4Etc^2}, \quad \epsilon_y = \frac{v_F - v_D}{2c} = 0,$$

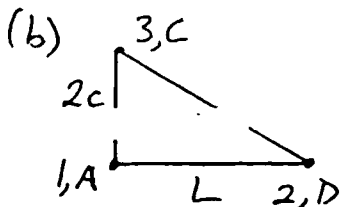
$$\gamma_{xy} = -\frac{u_D}{2c} + \frac{v_D}{L} + \frac{u_F}{2c} = \frac{u_F}{c} + \frac{v_D}{L} = -\frac{PL^2}{4Etc^3}$$

$$\text{With } \nu = 0, \quad \sigma_x = E\epsilon_x = \frac{3PL}{4tc^2}, \quad \sigma_y = E\epsilon_y = 0, \quad \tau_{xy} = \frac{E}{2}\gamma_{xy} = -\frac{PL^2}{8tc^3}$$

$$\text{Beam theory, top \& bottom: } |\sigma_x| = \frac{P(L-x)c}{I} = \frac{3P(L-x)}{2tc^2}$$

$$\text{Beam theory, neutral axis: } \tau_{xy} = \frac{3}{2} \frac{P}{2tc} = \frac{3P}{4tc}$$

$$\text{With } \sigma_x \text{ and } \tau_{xy} \text{ from FEA, } \frac{\tau_{xy}}{\sigma_x} = -\frac{L}{6c} \rightarrow -\infty \text{ as } \frac{L}{c} \rightarrow \infty$$



Obtain $[B]$ from Eq. 3.4-10

$$\{\epsilon\} = [B]\{d\} = \begin{bmatrix} -1/L & 0 & 1/L & 0 & 0 & 0 \\ 0 & -1/2c & 0 & 0 & 0 & 1/2c \\ -1/2c & -1/L & 0 & 1/L & 1/2c & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_D \\ v_D \\ 0 \\ 0 \end{Bmatrix}$$

$$\epsilon_x = \frac{u_D}{L} = \frac{3PL}{4Etc^2}, \quad \epsilon_y = 0, \quad \gamma_{xy} = \frac{v_D}{L} = \frac{PL^2}{2Etc^3}$$

$$\text{With } \nu = 0, \quad \sigma_x = E\epsilon_x = \frac{3PL}{4tc^2}, \quad \sigma_y = E\epsilon_y = 0, \quad \tau_{xy} = \frac{E}{2}\gamma_{xy} = \frac{PL^2}{4tc^3}$$

Beam theory: as in part (a)

$$\text{With } \sigma_x \text{ and } \tau_{xy} \text{ from FEA, } \frac{\tau_{xy}}{\sigma_x} = \frac{L}{3c} \rightarrow \infty \text{ as } \frac{L}{c} \rightarrow \infty$$

3.4-3

(a) The only nonzero d.o.f. are u_3 and v_3 . From Eq. 3.4-10,

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = [B] \{d\} = \begin{bmatrix} 0 & 0 \\ 0 & 1/a \\ 1/a & 0 \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix} \quad \text{For } v=0,$$

$$[k] = Et \frac{a^2}{2} \begin{bmatrix} 0 & 0 & 1/a \\ 0 & 1/a & 0 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$[k] = \frac{Et}{2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1/2 & 0 \end{bmatrix} = \frac{Et}{2} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) 4 triangles each resist P with ϵ_y stiffness $Et/4$
 4 triangles each resist P with γ_{xy} stiffness $Et/2$

Net stiffness = $4\left(\frac{Et}{4} + \frac{Et}{2}\right) = 3Et$, so $v = \frac{P}{3Et}$
 at center node (independent of a).

3.4-4 With u_1 and v_1 the only nonzero d.o.f., $\epsilon_x = 0$, $\epsilon_y = \frac{v_1}{b}$, $\gamma_{xy} = \frac{u_1}{b}$

$$\{\underline{\epsilon}\} = [\underline{B}]\{\underline{d}\} = \frac{1}{b} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} \quad \text{For } v=0,$$

$$[\underline{k}] = E(abt) \frac{1}{b^2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$[\underline{k}] = \frac{Eat}{b} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1/2 & 0 \end{bmatrix} = \frac{Eat}{b} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

3.5-1

(a) Let $\phi = u$ or v . ϕ depends only on y . Assume

$$\phi = c_1 + c_2 y + c_3 y^2 \quad (a)$$

Substitute:

$$\phi = 0 \text{ at } y = 0, \therefore c_1 = 0$$

$$\phi = 0 \text{ at } y = b, \quad 0 = c_2 b + c_3 b^2$$

$$\phi = 1 \text{ at } y = 2b, \quad 1 = c_2(2b) + c_3(4b^2)$$

$$\text{Solve: } c_2 = -\frac{1}{2b}, \quad c_3 = \frac{1}{2b^2}$$

$$\text{Eq. (a) becomes } \phi = \underbrace{-\frac{y}{2b} + \frac{y^2}{2b^2}}_{N_3}$$

$$(b) \text{ On } y=0, N_4 = 1 - \left(\frac{x}{a}\right)^2$$

Zero at $x = \pm a$, unity at $x = 0$

$$\text{On } y=b, N_4 = -\left(\frac{x}{a}\right)^2 + \frac{1}{4}$$

Zero at $x = \pm(a/2)$

$$\text{At } y = 2b, N_4 = 1 - 2 + 1 = 0$$

$$(c) \text{ Here } u = N_3 u_3 + N_4 u_4, \quad v = N_3 v_3 + N_4 v_4$$

$$\epsilon_x = \frac{\partial N_3}{\partial x} u_3 + \frac{\partial N_4}{\partial x} u_4 = -\frac{2x}{a^2} u_4$$

$$\epsilon_y = \frac{\partial N_3}{\partial y} v_3 + \frac{\partial N_4}{\partial y} v_4$$

$$= \left(-\frac{1}{2b} + \frac{y}{b^2}\right) v_3 + \left[-\frac{1}{b} + \frac{y}{2b^2}\right] v_4$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = (---) u_3 + [---] u_4 - \frac{2x}{a^2} v_4$$

3.6-1

$$\epsilon_x = a_2 + a_4 y, \quad \epsilon_y = a_7 + a_8 x$$

$$\gamma_{xy} = a_3 + a_4 x + a_6 + a_8 y$$

$$\sigma_{x,x} = \frac{E}{1-\nu^2} (\epsilon_x + \nu \epsilon_y)_{,x} = \frac{E}{1-\nu^2} (\nu a_8)$$

$$\sigma_{y,y} = \frac{E}{1-\nu^2} (\epsilon_y + \nu \epsilon_x)_{,y} = \frac{E}{1-\nu^2} (\nu a_4)$$

$$\tau_{xy} = \frac{E}{2(1+\nu)} \gamma_{xy} \quad \tau_{xy,x} = \frac{E}{2(1+\nu)} a_4$$

$$\text{Subs. into} \quad \tau_{xy,y} = \frac{E}{2(1+\nu)} a_8$$

$$\left. \begin{aligned} \sigma_{x,x} + \tau_{xy,y} + F_x &= 0 \\ \sigma_{y,y} + \tau_{xy,x} + F_y &= 0 \end{aligned} \right\} \text{yields} \begin{cases} C E a_8 + F_x = 0 \\ C E a_4 + F_y = 0 \end{cases}$$

$$\text{where } C = \frac{1}{1-\nu^2} + \frac{1}{2(1+\nu)} = \frac{1}{2(1-\nu)}$$

Eqs. satisfied if $a_4 = a_8 = 0$ when $F_x = F_y = 0$
or if particular values for F_x & F_y const.

3.6-2

$$(a) \quad \{ \underline{d} \} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} a_1 - a_2 a - a_3 b \\ a_4 - a_5 a - a_6 b \\ a_1 + a_2 a - a_3 b \\ a_4 + a_5 a - a_6 b \\ a_1 + a_2 a + a_3 b \\ a_4 + a_5 a + a_6 b \\ a_1 - a_2 a + a_3 b \\ a_4 - a_5 a + a_6 b \end{Bmatrix}$$

Eqs. 3.6-4 and 3.6-5 show that

$$[N] \{ \underline{d} \} = \begin{Bmatrix} a_1 + a_2 x + a_3 y \\ a_4 + a_5 x + a_6 y \end{Bmatrix}$$

$$(b) \quad \epsilon_x = \frac{1}{4ab} \begin{bmatrix} -(b-y) & 0 & b-y & 0 & b+y & 0 & -(b+y) & 0 \end{bmatrix} \{ \underline{d} \}$$

$$= a_2$$

$$\epsilon_y = \frac{1}{4ab} \begin{bmatrix} 0 & -(a-x) & 0 & -(a+x) & 0 & a+x & 0 & a-x \end{bmatrix} \{ \underline{d} \}$$

$$= a_6$$

$$\gamma_{xy} = \frac{1}{4ab} \begin{bmatrix} -(a-x) & -(b-y) & -(a+x) & b-y \\ a+x & b+y & a-x & -(b+y) \end{bmatrix} \{ \underline{d} \}$$

$$= a_3 + a_5$$

3.6-3 Beam theory will give $u_F = -u_D$, $v_F = v_D$

(a) From Eq. 3.6-6,

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{4ab} \begin{bmatrix} \cdots & b-y & 0 & b+y & 0 & \cdots \\ \cdots & 0 & -(a+x) & 0 & a+x & \cdots \\ \cdots & -(a+x) & b-y & a+x & b+y & \cdots \end{bmatrix} \begin{Bmatrix} u_D \\ v_D \\ -u_D \\ v_D \\ \vdots \end{Bmatrix}$$

From beam theory, Prob. 3.4-1, $u_D = -\frac{3ML}{2Etc^2}$, $v_D = -\frac{3ML^2}{4Etc^3}$

Hence

$$\epsilon_x = \frac{3MLy}{4Etabc^2}, \quad \epsilon_y = 0, \quad \gamma_{xy} = \frac{3ML(a+x)}{4abEtc^2} - \frac{3ML^2}{8aEtc^3}$$

$$\text{For } \nu = 0, \quad \sigma_x = E\epsilon_x = \frac{3MLy}{4tabc^2}, \quad \sigma_y = E\epsilon_y = 0,$$

$$\tau_{xy} = \frac{E}{2} \gamma_{xy} = \frac{3ML(a+x)}{8abtc^2} - \frac{3ML^2}{16atc^3}$$

$$\text{But } L = 2a \text{ and } c = b, \text{ so } \sigma_x = \frac{3My}{2tb^3}, \quad \sigma_y = 0, \quad \tau_{xy} = \frac{3Mx}{4b^3t}$$

(in the coordinate system of Fig. 3.6-1). In this system, $\sigma_x = \frac{My}{I} = \frac{3My}{2tb^3}$, $\sigma_y = \tau_{xy} = 0$ according to beam theory.

$$(b) \text{ From beam theory, Prob. 3.4-2, } u_D = \frac{3PL^2}{4Etc^2}, \quad v_D = \frac{PL^3}{2Etc^3}$$

$$\text{Hence } \epsilon_x = -\frac{3PL^2y}{8abEtc^2}, \quad \epsilon_y = 0, \quad \gamma_{xy} = \frac{PL^3}{4aEtc^3} - \frac{3PL^2(a+x)}{8abEtc^2}$$

$$\text{For } \nu = 0, \quad \sigma_x = E\epsilon_x = -\frac{3PL^2y}{8abtc^2}, \quad \sigma_y = E\epsilon_y = 0,$$

$$\tau_{xy} = \frac{E}{2} \gamma_{xy} = \frac{PL^3}{8atc^3} - \frac{3PL^2(a+x)}{16abtc^2}$$

$$\text{But } L = 2a \text{ and } c = b, \text{ so } \sigma_x = \frac{3Pay}{2tb^3}, \quad \sigma_y = 0, \quad \tau_{xy} = \frac{Pa}{4tb^3}(a-3x)$$

(in the coordinate system of Fig. 3.6-1). In this system,

$$\sigma_x = \frac{My}{I} = \frac{P(a-x)y}{t(2b)^3/12} = \frac{3P(a-x)y}{2tb^3}, \quad \sigma_y = 0, \text{ and}$$

$$\tau_{xy} = \frac{3}{2} \frac{P}{2tb} = \frac{3P}{4tb} \quad \text{according to beam theory.}$$

3.6-4

(a) $K_{48,39}$ is associated with v_{24} and u_{20} . Since node 24 is not in element j , el. j does not contribute to $K_{48,39}$.

(b) $K_{37,37} \leftarrow k_{1,1}$ of el. j . $[E]$ is diagonal (since $\nu = 0$), $[E] = 10^7 \begin{bmatrix} 1 & 1 & \frac{1}{2} \end{bmatrix}$. We need only col. 1 of $[B]$, Eq. 3.6-6

$$k_{1,1} = \frac{1}{16} \int_{-1}^1 \int_{-1}^1 \begin{Bmatrix} -(1-y) \\ 0 \\ -(1-x) \end{Bmatrix}^T \begin{Bmatrix} -(1-y) \\ 0 \\ -\frac{1}{2}(1-x) \end{Bmatrix} 10^7(1) dx dy$$

After integration, $k_{1,1} = 10^7/2$.

(c) $K_{59,61} \leftarrow k_{3,5}$ of el. j . Need cols. 3 & 5 of $[B]$, Eq. 3.6-6. As in (b),

$$k_{3,5} = \frac{1}{16} \int_{-1}^1 \int_{-1}^1 \begin{Bmatrix} 1-y \\ 0 \\ -(1+x) \end{Bmatrix}^T \begin{Bmatrix} 1+y \\ 0 \\ \frac{1}{2}(1+x) \end{Bmatrix} 10^7(1) dx dy$$

After integration, $k_{3,5} = 0$

3.6-5

Method 1: Evaluate $u = a_1 + a_2 x + a_3 y + a_4 xy$ at nodes, solve for a 's, gather coefficients of u_1, u_2, u_3, u_4 .

$$u_1 = a_1 \quad \text{Node 1}$$

$$u_2 = u_1 + a_2(2a); a_2 = \frac{u_2 - u_1}{2a} \quad \text{Node 2}$$

$$u_4 = u_1 + a_3(2b); a_3 = \frac{u_4 - u_1}{2b} \quad \text{Node 4}$$

$$u_3 = u_1 + \frac{u_2 - u_1}{2a} 2a + \frac{u_4 - u_1}{2b} 2b + a_4(2a)(2b)$$

$$\text{from which } a_4 = \frac{u_1 - u_2 + u_3 - u_4}{4ab}$$

$$u = u_1 + \frac{u_2 - u_1}{2a} x + \frac{u_4 - u_1}{2b} y + \frac{u_1 - u_2 + u_3 - u_4}{4ab} xy$$

$$u = \left(1 - \frac{x}{2a} - \frac{y}{2b} + \frac{xy}{4ab}\right) u_1 + \left(\frac{x}{2a} - \frac{xy}{4ab}\right) u_2 + \left(\frac{xy}{4ab}\right) u_3 + \left(\frac{y}{2b} - \frac{xy}{4ab}\right) u_4$$

Coefficients of the u_i are the N_i .

Method 2: Take the product of one-dimensional N_i (as for a bar element).

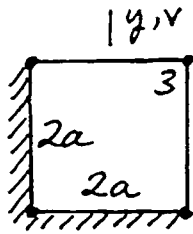
$$N_1 = \frac{2a-x}{2a} \frac{2b-y}{2b} \quad N_2 = \frac{x}{2a} \frac{2b-y}{2b}$$

$$N_3 = \frac{x}{2a} \frac{y}{2b} \quad N_4 = \frac{2a-x}{2a} \frac{y}{2b}$$

Method 3: In Eqs. 3.6-4, replace x by $x-a$ and y by $y-b$.

3.6-6

Consider the lower-left element of the four-element structure.



In the assembled structure, v_3 is the only nonzero d.o.f., so we need only this one stiffness coefficient.

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = [\underline{B}]\{d\} = \frac{1}{4a^2} \begin{Bmatrix} 0 \\ a+x \\ a+y \end{Bmatrix} v_3 \quad (\text{from Eq. 3.6-6})$$

$$k = \frac{1}{16a^4} \int_{-a}^a \int_{-a}^a \begin{bmatrix} 0 & a+x & a+y \end{bmatrix} \begin{bmatrix} E \\ E \\ E/2 \end{bmatrix} \begin{Bmatrix} 0 \\ a+x \\ a+y \end{Bmatrix} t \, dx \, dy$$

$$k = \frac{Et}{16a^4} \int_{-a}^a \int_{-a}^a \left[(a+x)^2 + \frac{(a+y)^2}{2} \right] dx \, dy = \frac{Et}{16a^4} \left[\frac{16a^4}{3} + \frac{8a^4}{3} \right] = \frac{Et}{2}$$

$$\text{For the four-element structure, } v_{\text{center}} = \frac{P/4}{Et/2} = \frac{P}{2Et}$$

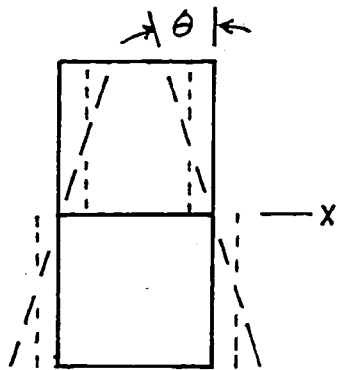
3.6-7

Assume that beam theory would provide almost the exact tip deflection, and recall Eq. 3.6-11:

$$\frac{\theta_{el}}{\theta_b} = \frac{1-\nu^2}{1 + \frac{1-\nu}{2} \left(\frac{a}{b}\right)^2}$$

(a) Expect the 2nd mesh to be better, since a/b is half as large in the 2nd mesh

(b) Two elements of the first mesh, with bending deformation:



Smaller dashed lines represent deformation due to ϵ_x in elements. Assume this is exact. Remainder comes from rotation of element faces and suffers from parasitic shear: with $\nu=0$,

$$\frac{\theta_{el}}{\theta_b} = \frac{1}{1 + \frac{1}{2}(1)} = \frac{2}{3}$$

Assume that γ_{xy} and ϵ_x contribute equally, therefore estimate error as the average:

$$e \approx \frac{1}{2} \left[0 + \left(1 - \frac{2}{3}\right) \right] \approx 17\%$$

For the second mesh, $\frac{a}{b} = \frac{1}{2}$, so with $\nu=0$

$$\frac{\theta_{el}}{\theta_b} = \frac{1}{1 + \frac{1}{2} \left(\frac{1}{2}\right)^2} = 0.89 \quad \text{i.e. } e \approx 11\%$$

3.7-1

$$\phi_{152} = \frac{-x(a-x)}{2a^2} \phi_1 + \frac{(a-x)(a+x)}{a^2} \phi_5 + \frac{x(a+x)}{2a^2} \phi_2$$

$$\phi_{43} = \frac{a-x}{2a} \phi_4 + \frac{a+x}{2a} \phi_3$$

$$\phi = \frac{b-y}{2b} \phi_{152} + \frac{b+y}{2b} \phi_{43} = \sum_{i=1}^5 N_i \phi_i$$

$$N_1 = \frac{-x(a-x)(b-y)}{4a^2b}, \quad N_2 = \frac{x(a+x)(b-y)}{4a^2b}$$

$$N_3 = \frac{(a+x)(b+y)}{4ab}, \quad N_4 = \frac{(a-x)(b+y)}{4ab}$$

$$N_5 = \frac{(a^2-x^2)(b-y)}{2a^2b}$$

3.7-2

$$(a) \quad \phi_{184} = \sum_{i=1}^3 \bar{N}_i * (\phi_i \text{ on } x = -a)$$

$$\phi_{597} = \sum_{i=1}^3 \bar{N}_i * (\phi_i \text{ on } x = 0)$$

$$\phi_{263} = \sum_{i=1}^3 \bar{N}_i * (\phi_i \text{ on } x = +a)$$

for which $\bar{N}_1, \bar{N}_2, \bar{N}_3$ come from Eq.

3.2-7 with $x \rightarrow y, x_1 \rightarrow -b, x_2 = 0, x_3 \rightarrow b$

$$\bar{N}_1 = \frac{-y(b-y)}{2b^2}, \quad \bar{N}_2 = \frac{(-b-y)(b-y)}{-b(b)} = \frac{b^2-y^2}{b^2}$$

$$\bar{N}_3 = \frac{(-b-y)(-y)}{-2b(-b)} = \frac{y(b+y)}{2b^2}. \text{ In 9-node el.,}$$

$$\phi = \frac{-x(a-x)}{2a^2} \phi_{184} + \frac{a^2-x^2}{a^2} \phi_{597} + \frac{x(a+x)}{2a^2} \phi_{263}$$

$$\phi = N_a \phi_{184} + N_b \phi_{597} + N_c \phi_{263}$$

$$\phi = N_a (\bar{N}_1 \phi_1 + \bar{N}_2 \phi_8 + \bar{N}_3 \phi_4) + N_b (\bar{N}_1 \phi_5 + \bar{N}_2 \phi_9 + \bar{N}_3 \phi_7) + N_c (\bar{N}_1 \phi_2 + \bar{N}_2 \phi_6 + \bar{N}_3 \phi_3)$$

$$\phi = \sum_{i=1}^9 N_i \phi_i \quad \text{where}$$

$$N_1 = N_a \bar{N}_1 = xy(a-x)(b-y)/4a^2b^2$$

$$N_2 = N_c \bar{N}_1 = -xy(a+x)(b-y)/4a^2b^2$$

$$N_3 = N_c \bar{N}_3 = xy(a+x)(b+y)/4a^2b^2$$

$$N_4 = N_a \bar{N}_3 = -xy(a-x)(b+y)/4a^2b^2$$

$$N_5 = N_b \bar{N}_1 = -(a^2-x^2)y(b-y)/2a^2b^2$$

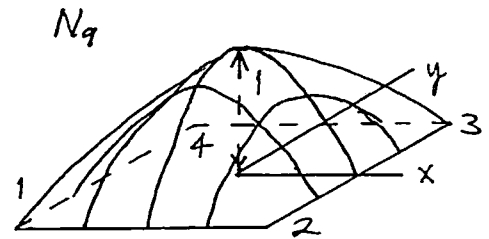
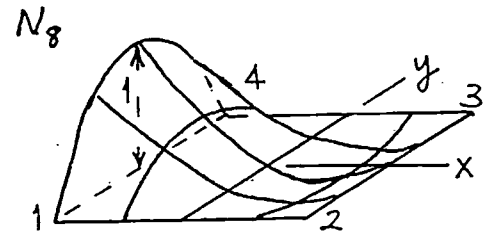
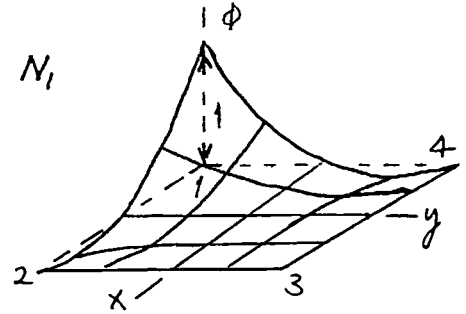
$$N_6 = N_c \bar{N}_2 = x(a+x)(b^2-y^2)/2a^2b^2$$

$$N_7 = N_b \bar{N}_3 = (a^2-x^2)y(b+y)/2a^2b^2$$

$$N_8 = N_a \bar{N}_2 = -x(a-x)(b^2-y^2)/2a^2b^2$$

$$N_9 = N_c \bar{N}_2 = (a^2-x^2)(b^2-y^2)/a^2b^2$$

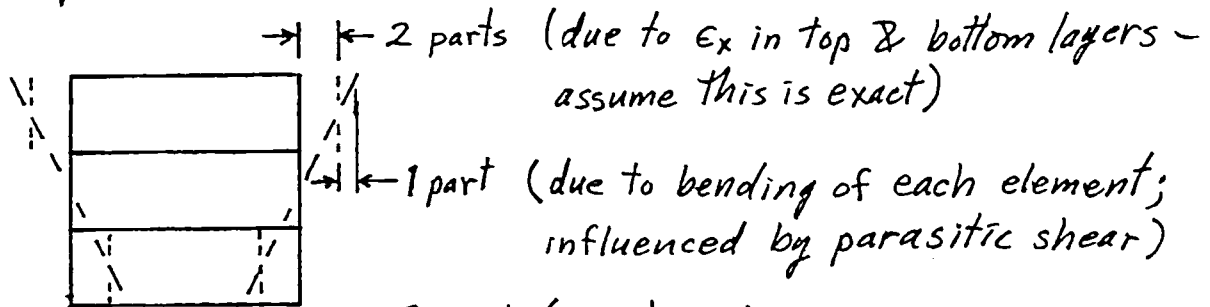
(b)



3.8-1 Similar to Problem 3.6-7. Again use Eq. 3.6-11 with $\nu = 0$ for simplicity. Best mesh to worst, with estimated error e :

(c) Aspect ratio $\frac{1}{3}$, $\frac{1}{1 + \frac{1}{2}(\frac{1}{3})^2} = 0.947$, $e \approx 1 - 0.947 = 5.3\%$.

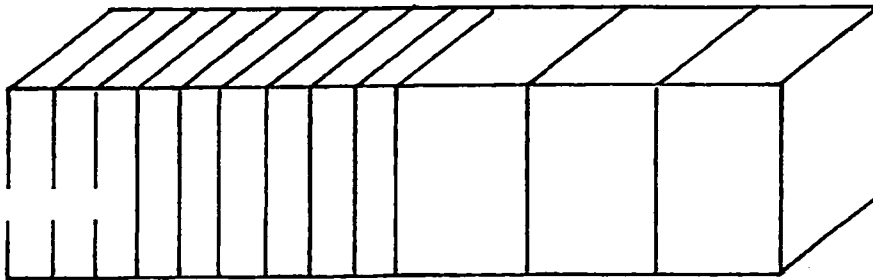
(b) Aspect ratio 3



$$\frac{2}{3} + \frac{1}{3} \left(\frac{1}{1 + \frac{1}{2}(3)^2} \right) = 0.727, e \approx 1 - 0.727 = 27\%$$

(c) Aspect ratio 1, $\frac{1}{1 + \frac{1}{2}(1)^2} = 0.667$, $e \approx 1 - 0.667 = 33\%$

A better 12-element mesh (for the given loading):



3.9-1

Rigid-body lateral translation $\bar{u}$: $\{\underline{d}\} = [\bar{u} \ 0 \ \bar{u} \ 0]^T$

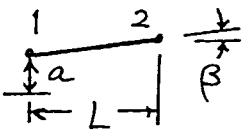
Rigid-body rotation through small angle θ about node 1:

$$\{\underline{d}\} = [0 \ \theta \ L\theta \ \theta]^T$$

In both cases, straightforward multiplication shows that

$[\underline{B}]\{\underline{d}\}$ and $[\underline{k}]\{\underline{d}\}$ are both zero.

3.9-2

(a)  $w = a + \beta x$ gives
 $w_1 = a, \theta_1 = \beta$
 $w_2 = a + \beta L, \theta_2 = \beta$

Subs. into given field; get $w = a + \beta x$
 OK

(b) From given field, $w_{,xx} = (\theta_2 - \theta_1)/L$
 By beam theory: let $M_0 = \text{const. moment}$

$EI w_{,xx} = M_0$ (a) Eliminate M_0
 $\int_{\theta_1}^{\theta_2} EI dw_{,x} = \int_0^L M_0 dx$ between (a) &
(b); get
 $EI(\theta_2 - \theta_1) = M_0 L$ (b) $w_{,xx} = (\theta_2 - \theta_1)/L$

(c) $w_{,xx} = [\underline{B}]\{\underline{d}\} = \left[0, -\frac{1}{L}, 0, \frac{1}{L}\right] [w_1, \theta_1, w_2, \theta_2]^T$

$[\underline{k}] = \int_0^L [\underline{B}]^T [\underline{B}] EI dx = \frac{EI}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

This $[\underline{k}]$ offers no
 resistance to w_1 & w_2
 and develops no nodal forces in response
 to θ_1 & θ_2 . $[\underline{k}]$ has rank 1.

3.9-3

(a)

Evaluate $u = [1, x^2] \{a\}$ at nodes.

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & L^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}, \quad \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{1}{L^2} \begin{bmatrix} L^2 & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [\underline{A}]^{-1} \{\underline{u}\}$$

$$u = [1, x^2] [\underline{A}]^{-1} \{\underline{u}\} = \begin{bmatrix} \frac{L^2 - x^2}{L^2} & \frac{x^2}{L^2} \end{bmatrix} \{\underline{u}\}$$

$$\epsilon_x = \begin{bmatrix} -\frac{2x}{L^2} & \frac{2x}{L^2} \end{bmatrix} \{\underline{u}\} = [\underline{B}] \{\underline{u}\}$$

$$[\underline{k}] = \int_0^L [\underline{B}]^T [\underline{B}] AE dx = \frac{4AE}{3L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Too large by factor $\frac{4}{3}$. Source is defective u : no x^1 term present, so state of constant ϵ_x is not possible.

(b)

Evaluate $u = [x, x^2] \{a\}$ at nodes.

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} -\frac{L}{2} & \frac{L^2}{4} \\ \frac{L}{2} & \frac{L^2}{4} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}, \quad \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{1}{L^2} \begin{bmatrix} -L & L \\ 2 & 2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [\underline{A}]^{-1} \{\underline{u}\}$$

$$u = [x, x^2] [\underline{A}]^{-1} \{\underline{u}\} = \begin{bmatrix} \frac{2x^2 - Lx}{L^2} & \frac{2x^2 + Lx}{L^2} \end{bmatrix} \{\underline{u}\}$$

$$\epsilon_x = \begin{bmatrix} \frac{4x - L}{L^2} & \frac{4x + L}{L^2} \end{bmatrix} \{\underline{u}\} = [\underline{B}] \{\underline{u}\}$$

$$[\underline{k}] = \int_{-L/2}^{L/2} [\underline{B}]^T [\underline{B}] AE dx = \frac{AE}{3L} \begin{bmatrix} 7 & 1 \\ 1 & 7 \end{bmatrix}$$

$\dots$ when $\{d\} = \{1, 1\}$ (rigid body translation). Source of trouble is defective u : no constant term present.

3.9-4

Exact ans.: $u_2 = \frac{PL}{AE}$, $u_3 = 2 \frac{PL}{AE}$

(a) $\frac{4AE}{3L} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}, \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \frac{PL}{AE} \begin{Bmatrix} 3/4 \\ 3/2 \end{Bmatrix}$

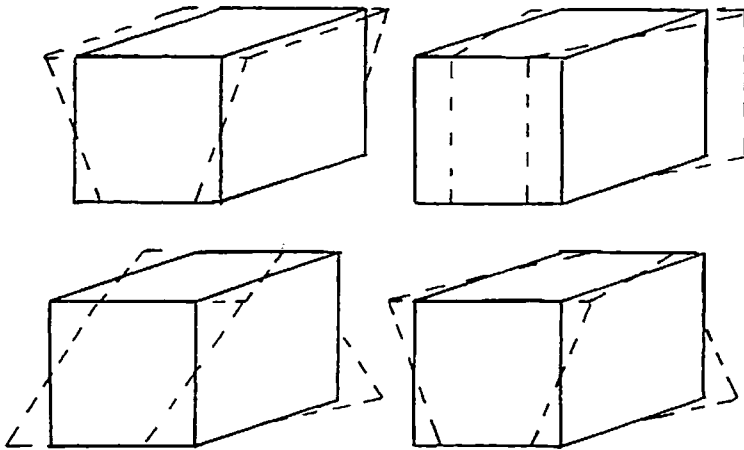
$\sigma_{x=0} = E \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = 0$ (Exact σ_x is P/A for all x)

(b) $\frac{AE}{3L} \begin{bmatrix} 14 & 1 \\ 1 & 7 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}, \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \frac{PL}{AE} \begin{Bmatrix} -3/97 \\ 42/97 \end{Bmatrix}$

$\sigma_{x=0} = E \begin{bmatrix} \frac{-3L}{L^2} & \frac{-L}{L^2} \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = -\frac{1}{L} \frac{3PL}{97AE} = \frac{3P}{97A}$

at $x=0$ in structure; at $x=-\frac{L}{2}$ in element.

3.9-5



With the node numbering of Fig. 3.8 1a, x-direction nodal d.o.f. of the foregoing states are, with c a constant,

$$\begin{Bmatrix} c \\ c \\ -c \\ -c \\ -c \\ -c \\ c \\ c \end{Bmatrix}$$

$$\begin{Bmatrix} c \\ -c \\ -c \\ c \\ -c \\ c \\ c \\ -c \end{Bmatrix}$$

$$\begin{Bmatrix} -c \\ -c \\ c \\ -c \\ c \\ c \\ c \\ c \end{Bmatrix}$$

$$\begin{Bmatrix} c \\ -c \\ c \\ -c \\ -c \\ c \\ -c \\ c \end{Bmatrix}$$

3.10-1

$$\text{Let } a_1 = \frac{1}{8}(x_2 - x_3)$$

$$a_2 = \frac{1}{8}(x_3 - x_1)$$

$$a_3 = \frac{1}{8}(x_1 - x_2)$$

$$b_1 = \frac{1}{8}(y_3 - y_2)$$

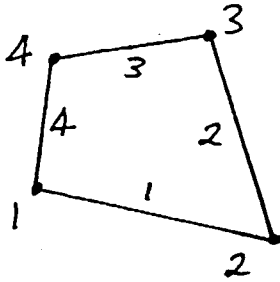
$$b_2 = \frac{1}{8}(y_1 - y_3)$$

$$b_3 = \frac{1}{8}(y_2 - y_1)$$

$$\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & -b_3 & \frac{1}{2} & 0 & b_3 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -a_3 & 0 & \frac{1}{2} & a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & -b_1 & \frac{1}{2} & 0 & b_1 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & -a_1 & 0 & \frac{1}{2} & a_1 \\ \frac{1}{2} & 0 & b_2 & 0 & 0 & 0 & \frac{1}{2} & 0 & -b_2 \\ 0 & \frac{1}{2} & a_2 & 0 & 0 & 0 & 0 & \frac{1}{2} & -a_2 \end{bmatrix}}_{[\tilde{T}]} \begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ u_3 \\ v_3 \\ w_3 \end{bmatrix}$$

3.10-2

Consider e.g. a quadrilateral element, and apply Eq. 3.10-1 to each side.



$$\frac{\delta_{m1}}{L_1} = \frac{1}{8}(\omega_2 - \omega_1)$$

$$\frac{\delta_{m2}}{L_2} = \frac{1}{8}(\omega_3 - \omega_2)$$

$$\frac{\delta_{m3}}{L_3} = \frac{1}{8}(\omega_4 - \omega_3)$$

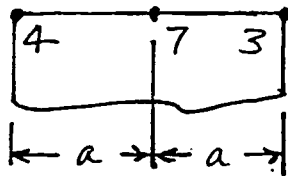
$$\frac{\delta_{m4}}{L_4} = \frac{1}{8}(\omega_1 - \omega_4)$$

Add; thus

$$\sum_{i=1}^4 \frac{\delta_{mi}}{L_i} = 0$$

3.11-1

Eq. 3.11-6:
$$\begin{Bmatrix} F_4 \\ F_7 \\ F_3 \end{Bmatrix} = \underbrace{\frac{a}{15} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}}_{[H]} \begin{Bmatrix} q_4 \\ q_7 \\ q_3 \end{Bmatrix}$$



Unit thickness

(a)
$$\begin{Bmatrix} F_4 \\ F_7 \\ F_3 \end{Bmatrix} = [H] \begin{Bmatrix} \sigma \\ 0 \\ -\sigma \end{Bmatrix} = \frac{\sigma a}{3} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$$

Couple-moment: $M = F_4(2a) = \frac{2\sigma a^2}{3}$

Flexure formula:

$$M = \frac{\sigma I}{c} = \frac{\sigma (2a)^3 / 12}{a} = \frac{2\sigma a^2}{3}$$

(b)
$$\begin{Bmatrix} F_4 \\ F_7 \\ F_3 \end{Bmatrix} = [H] \begin{Bmatrix} 0 \\ \sigma/2 \\ \sigma \end{Bmatrix} = \frac{\sigma a}{3} \begin{Bmatrix} 0 \\ 2 \\ 1 \end{Bmatrix}$$

Moment: $M = 2aF_3 + aF_7 = \frac{4\sigma a^2}{3}$

Flexure formula, for section 4a units deep:

$$M = \frac{\sigma I}{c} = \frac{\sigma (4a)^3 / 12}{2a} = \frac{8\sigma a^2}{3}$$

Contribution of half the section is

$$\frac{M}{2} = \frac{4\sigma a^2}{3}$$

(c)
$$\begin{Bmatrix} F_4 \\ F_7 \\ F_3 \end{Bmatrix} = [H] \begin{Bmatrix} 0 \\ \tau \\ 0 \end{Bmatrix} = \frac{\tau a}{15} \begin{Bmatrix} 2 \\ 16 \\ 2 \end{Bmatrix}$$

Shear force: $F_4 + F_7 + F_3 = \frac{4\tau a}{3}$

Beam theory (parabolic distribution):

$$\tau = \frac{3}{2} \frac{V}{2a}, \quad V = \frac{4\tau a}{3}$$

3.11-2

With $[N]$ from Eq. 3.11-5, and constant q ,

$$\begin{aligned} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} &= \int_{-a}^a [N] q dx = \int_{-a}^a \frac{1}{2a^2} \begin{Bmatrix} x(x-a) \\ 2(a^2-x^2) \\ x(x+a) \end{Bmatrix} q dx \\ &= \frac{q}{2a^2} \begin{Bmatrix} \frac{x^3}{3} - \frac{ax^2}{2} \\ 2a^2x - \frac{2x^3}{3} \\ \frac{x^3}{3} + \frac{ax^2}{2} \end{Bmatrix} \bigg|_{-a}^a = \frac{q}{2a^2} \begin{Bmatrix} 2a^3/3 \\ 8a^3/3 \\ 2a^3/3 \end{Bmatrix} = q \begin{Bmatrix} a/3 \\ 4a/3 \\ a/3 \end{Bmatrix} = F \begin{Bmatrix} 1/6 \\ 2/3 \\ 1/6 \end{Bmatrix} \end{aligned}$$

where $F = 2qa$

3.11-3

$$M_1 \delta \theta_{z_1} = \int_0^L v q dx \quad \text{where } v = N_2 \delta \theta_{z_1} \quad \text{and } q \text{ is constant}$$

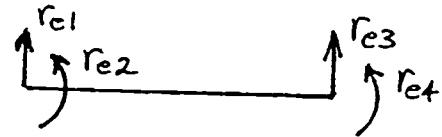
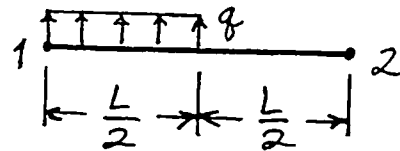
$$\text{Hence } M_1 = q \int_0^L \left(x - \frac{2x^2}{L} + \frac{x^3}{L^2} \right) dx = q L^2 \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) = \frac{q L^2}{12}$$

3.11-4

(a)

$$\{r_e\} = \int_0^{L/2} [N]^T dx = q \begin{Bmatrix} x - \frac{x^3}{L^2} + \frac{x^4}{2L^3} \\ \frac{x^2}{2} - \frac{2x^3}{3L} + \frac{x^4}{4L^2} \\ \frac{x^3}{L^2} - \frac{x^4}{2L^3} \\ -\frac{x^3}{3L} + \frac{x^4}{4L^2} \end{Bmatrix}_0^{L/2}$$

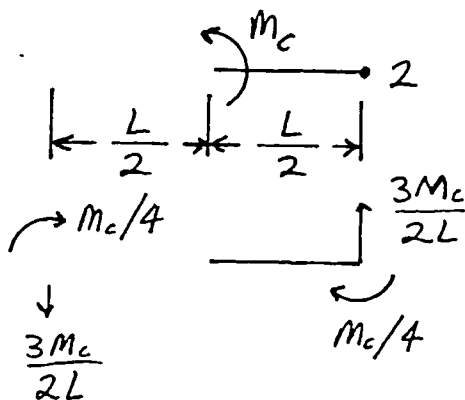
$$\{r_e\} = q \begin{Bmatrix} 0.40625L \\ 0.05729L^2 \\ 0.09375L \\ -0.02604L^2 \end{Bmatrix} \quad \begin{aligned} \sum F_z &= \frac{qL}{2} \checkmark \\ \sum M_o &= r_{e3}L + r_{e2} + r_{e4} \\ &= \frac{qL^2}{8} \checkmark \end{aligned}$$



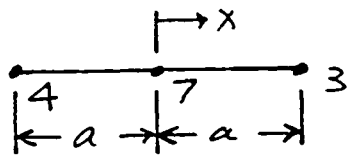
(b) $\{r_e\} = \left[\frac{dN}{dx} \right]_x^T M_c$ where $\left[\frac{dN}{dx} \right]_x^T$ is evaluated at $x = \frac{L}{2}$

From N_i in Fig. 3.2-4 we obtain

$$\begin{Bmatrix} -\frac{6x}{L^2} + \frac{6x^2}{L^3} \\ 1 - \frac{4x}{L} + \frac{3x^2}{L^2} \\ \frac{6x}{L^2} - \frac{6x^2}{L^3} \\ -\frac{2x}{L} + \frac{3x^2}{L^2} \end{Bmatrix}_{x=\frac{L}{2}} = \begin{Bmatrix} -\frac{6}{2L} + \frac{6}{4L} \\ 1 - 2 + \frac{3}{4} \\ \frac{6}{2L} - \frac{6}{4L} \\ -1 + \frac{3}{4} \end{Bmatrix} = \begin{Bmatrix} -\frac{3}{2L} \\ -\frac{1}{4} \\ \frac{3}{2L} \\ -\frac{1}{4} \end{Bmatrix}$$



3.11-5



$$v = [N_4 \quad N_7 \quad N_3]^T \begin{Bmatrix} v_4 \\ v_7 \\ v_3 \end{Bmatrix}$$

Apply Eq. 3.2-7: $N_4 = \frac{-x(a-x)}{a(2a)} = -\frac{x(a-x)}{2a^2}$

$$N_7 = \frac{(-a-x)(a-x)}{-a(a)} = \frac{a^2 - x^2}{a^2}$$

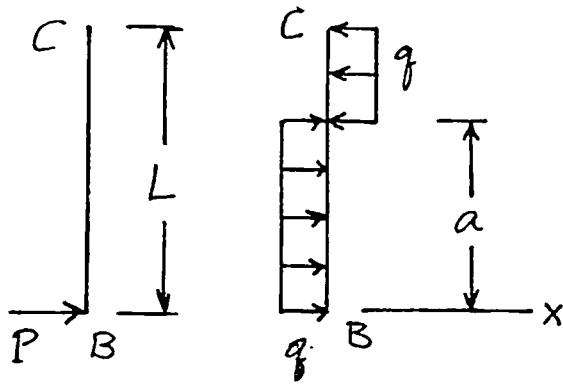
$$N_3 = \frac{(-a-x)(-x)}{-2a(-a)} = \frac{x(a+x)}{2a^2}$$

$$\begin{Bmatrix} F_4 \\ F_7 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} N_4 \\ N_7 \\ N_3 \end{Bmatrix}_{x=\frac{a}{2}} F = \begin{Bmatrix} -\frac{(a/2)^2}{2a^2} \\ \frac{a^2 - (a/2)^2}{a^2} \\ \frac{a/2(3a/2)}{2a^2} \end{Bmatrix} F = \begin{Bmatrix} -1/8 \\ 3/4 \\ 3/8 \end{Bmatrix} F$$

$$F_4 + F_7 + F_3 = F \quad \checkmark$$

$$\sum M_4 = \frac{3F}{4}a + \frac{3F}{8}(2a) = \frac{3Fa}{4} \quad \checkmark$$

3.11-6



$$\text{where } q = \frac{(1+\sqrt{2})P}{L}$$

$$a = \frac{L}{\sqrt{2}}$$

$$\text{x-direction force: } qa - q(L-a) = q(2a-L)$$

$$= \frac{(1+\sqrt{2})P}{L} \left(\frac{2L}{\sqrt{2}} - L \right)$$

$$= \frac{(1+\sqrt{2})(2-\sqrt{2})}{\sqrt{2}} P$$

$$= \frac{2+\sqrt{2}-2}{\sqrt{2}} P = P \quad \checkmark$$

$$\text{Moment about B: } qa \frac{a}{2} - q(L-a) \frac{L+a}{2} = \frac{q}{2} (2a^2 - L^2)$$

$$= \frac{q}{2} (L^2 - L^2) = 0 \quad \checkmark$$

3.11-7

Evaluate N_i of Eqs. 3.6-4 at Q ($x = \frac{a}{2}$, $y = 0$):

$$\{\tilde{r}_e\} = [N]_a^T \begin{Bmatrix} P_x \\ P_y \end{Bmatrix} = \frac{1}{4ab} \begin{bmatrix} ab/2 & 0 & 3ab/2 & 0 & 3ab/2 & 0 & ab/2 & 0 \\ 0 & ab/2 & 0 & 3ab/2 & 0 & 3ab/2 & 0 & ab/2 \end{bmatrix} \begin{Bmatrix} P_x \\ P_y \end{Bmatrix}$$

$$\{\tilde{r}_e\} = \frac{1}{8} \begin{bmatrix} 1 & 0 & 3 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 & 0 & 3 & 0 & 1 \end{bmatrix}^T \begin{Bmatrix} 8 \\ 6 \end{Bmatrix}$$

$$\{\tilde{r}_e\} = \begin{bmatrix} 1 & \frac{3}{4} & 3 & \frac{9}{4} & 3 & \frac{9}{4} & 1 & \frac{3}{4} \end{bmatrix}^T$$

3.11-8

Get N_i from Eq. 3.2-7. Let $a = \frac{L}{2}$. $B_i = \frac{d}{dx} N_i$

$$N_1 = -\frac{x(a-x)}{2a^2}$$

$$N_2 = \frac{a^2 - x^2}{a^2}$$

$$N_3 = \frac{x(a+x)}{2a^2}$$

$$B_1 = -\frac{a-2x}{2a^2}$$

$$B_2 = -\frac{2x}{a^2}$$

$$B_3 = \frac{a+2x}{2a^2}$$

$$\{\underline{r}_e\} = \int_{-a}^a [\underline{B}]^T [\underline{E}] \{\underline{\epsilon}_0\} dV = \int_{-a}^a \begin{Bmatrix} B_1 \\ B_2 \\ B_3 \end{Bmatrix} E \left(\alpha T_3 \frac{a+x}{2a} \right) A dx$$

$$\{\underline{r}_e\} = \frac{EA\alpha T_3}{4a^3} \int_{-a}^a \begin{Bmatrix} -(a-2x) \\ -4x \\ a+2x \end{Bmatrix} (a+x) dx = \frac{EA\alpha T_3}{4a^3} \int_{-a}^a \begin{Bmatrix} -a^2+x+2x^2 \\ -4ax-4x^2 \\ a^2+3x+2x^2 \end{Bmatrix} dx$$

$$\{\underline{r}_e\} = \frac{EA\alpha T_3}{4a^3} \begin{Bmatrix} -2a^3+4a^3/3 \\ -8a^3/3 \\ 2a^3+4a^3/3 \end{Bmatrix} = \frac{EA\alpha T_3}{4a^3} \begin{Bmatrix} -2a^3/3 \\ -8a^3/3 \\ 10a^3/3 \end{Bmatrix} = \frac{EA\alpha T_3}{6} \begin{Bmatrix} -1 \\ -4 \\ 5 \end{Bmatrix}$$

3.11-9

$$\{r_e\} = - \int [B]^T \{\sigma_0\} dV$$

To show $\sum r_{ei} = 0$, we ask if $[I]\{r_e\} = 0$, where $[I]$ is a row vector filled with 1's. We get $\sum r_{ei} = 0$ if $[I]\{B_j\} = 0$, where $\{B_j\}$ is a column of $[B]^T$, for all j in $[B]^T$. Now $[B] = [\partial][N]$, so each row of $[B]$ (and each column of $[B]^T$) is a derivative of $[N]$. But $\sum N_i = 1$ for C^0 elements, so $\sum N_{i,x} = 0$.

3.11-10

$$[\underline{E}]\{\underline{\epsilon}_0\} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \alpha T_0 x \\ \alpha T_0 x \\ 0 \end{Bmatrix} = \frac{E \alpha T_0 x}{1+\nu} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$$

$$\{r_e\} = \iint [B]^T [\underline{E}]\{\underline{\epsilon}_0\} t dx dy, N_i = \frac{(a \pm x)(b \pm y)}{4ab}$$

$N_{i,x}$ indep. of x but $\{\underline{\epsilon}_0\}$ linear in x , so integration w.r.t. x with limits $-a$ to a yields zero x -direction nodal loads.

$$\{r_e\}_{y\text{-dir.}} = \frac{E \alpha T_0}{(1+\nu)4ab} \int_{-b}^b \int_{-a}^a \begin{Bmatrix} -(a-x) \\ -(a+x) \\ (a+x) \\ (a-x) \end{Bmatrix} x dx dy$$

$$\{r_e\}_{y\text{-dir.}} = \frac{E a^2 \alpha T_0}{3(1+\nu)} \begin{Bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{Bmatrix} = \begin{Bmatrix} F \\ -F \\ F \\ -F \end{Bmatrix} \quad \begin{array}{cc} \downarrow F & F \uparrow \\ 4 & 3 \\ \uparrow F & F \downarrow \\ 1 & 2 \end{array} \quad \text{(as applied to structure nodes)}$$

(a) First the nodal loads, all 3 cases.

$$[N] = \begin{bmatrix} \frac{L-s}{L} & \frac{s}{L} \end{bmatrix}, F = \frac{c}{A}(L_T - x)$$

$$\{r_e\} = \int_0^L [N]^T F dV \text{ where } dV = A ds$$

$$\text{One element: } s=x, L_T=L, \{r_e\} = \frac{cL^2}{6} \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$$

$$\text{Two elements: in the first, } s=x, L_T=2L \\ \text{in the second, } x=L+s, L_T=2L$$

$$\{r_e\}_1 = \frac{cL^2}{6} \begin{Bmatrix} 5 \\ 4 \end{Bmatrix}, \{r_e\}_2 = \frac{cL^2}{6} \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$$

$$\text{Three elements: } L_T=3L; \text{ in the respec-} \\ \text{tive elements, } x=s, x=L+s, x=2L+s.$$

$$\{r_e\}_1 = \frac{cL^2}{6} \begin{Bmatrix} 8 \\ 7 \end{Bmatrix}, \{r_e\}_2 = \frac{cL^2}{6} \begin{Bmatrix} 5 \\ 4 \end{Bmatrix}, \{r_e\}_3 = \frac{cL^2}{6} \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$$

(b) • One el. $\frac{AE}{L} u_2 = \frac{cL^2}{6}, u_2 = \frac{cL^3}{6AE}$
 $(L=L_T)$ $\sigma_x = E \frac{u_2}{L} = \frac{cL^2}{6A}$

• Two e/s. $\frac{AE}{L} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \frac{cL^2}{6} \begin{Bmatrix} 4+2 \\ 1 \end{Bmatrix}$
 $(L_T=2L)$

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \frac{cL^3}{6AE} \begin{Bmatrix} 7 \\ 8 \end{Bmatrix} = \frac{cL_T^3}{6EA} \begin{Bmatrix} 7/8 \\ 1 \end{Bmatrix} \text{ (exact)}$$

$$\sigma_{x1} = E \frac{u_2}{L} = \frac{7cL^2}{6A} = \frac{7}{24} \frac{cL_T^2}{A} = 0.2917 \frac{cL_T^2}{A}$$

$$\sigma_{x2} = E \frac{u_3 - u_2}{L} = \frac{1}{7} \sigma_{x1} = 0.0417 \frac{cL_T^2}{A}$$

• $\frac{AE}{L} \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \frac{cL^2}{6} \begin{Bmatrix} 7+5 \\ 4+2 \\ 1 \end{Bmatrix}$ 3 e/s.
 $(L_T=3L)$

$$\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \frac{cL^3}{6AE} \begin{Bmatrix} 19 \\ 26 \\ 27 \end{Bmatrix} \quad \sigma_{x1} = E \frac{u_2}{L} = 0.3519 \frac{cL_T^2}{A}$$

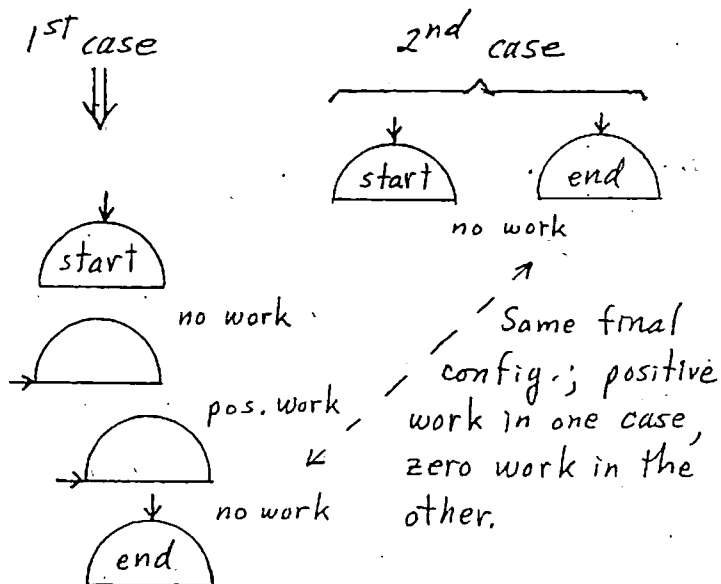
$$\sigma_{x2} = E \frac{u_3 - u_2}{L} = 0.1296 \frac{cL_T^2}{A}$$

$$\sigma_{x3} = E \frac{u_4 - u_3}{L} = 0.0185 \frac{cL_T^2}{A}$$

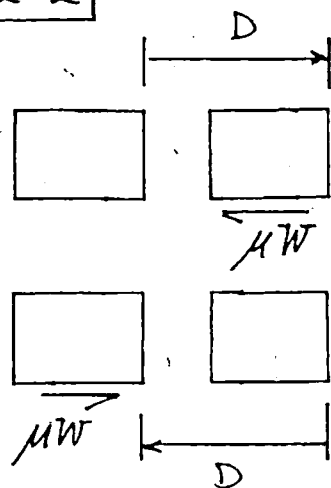
Exact stresses: multipliers of cL_T^2/A are:

$\frac{x}{L_T}$	0	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{5}{6}$	1
factor	.5	.3472	.2813	.1250	.0313	.0139	0
	left end			middle			right end

4.2-1



4.2-2



Work of friction force:

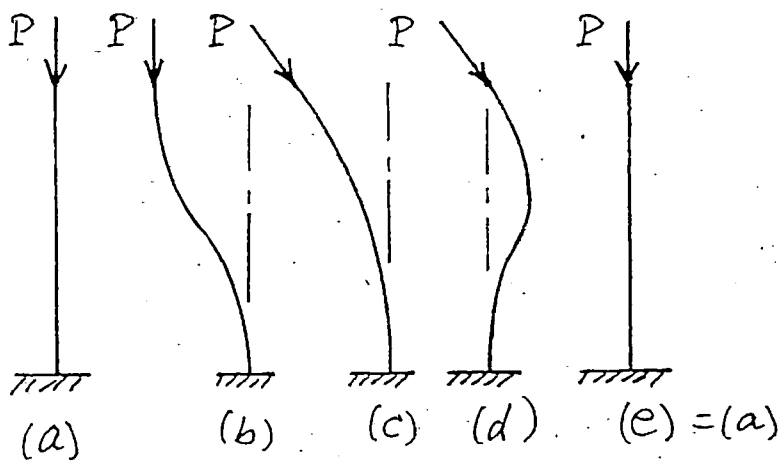
$$(-\mu W) D$$

$$\mu W (-D)$$

Net work of friction force is $-2\mu WD$
Not zero, so not conservative.

4.2-3

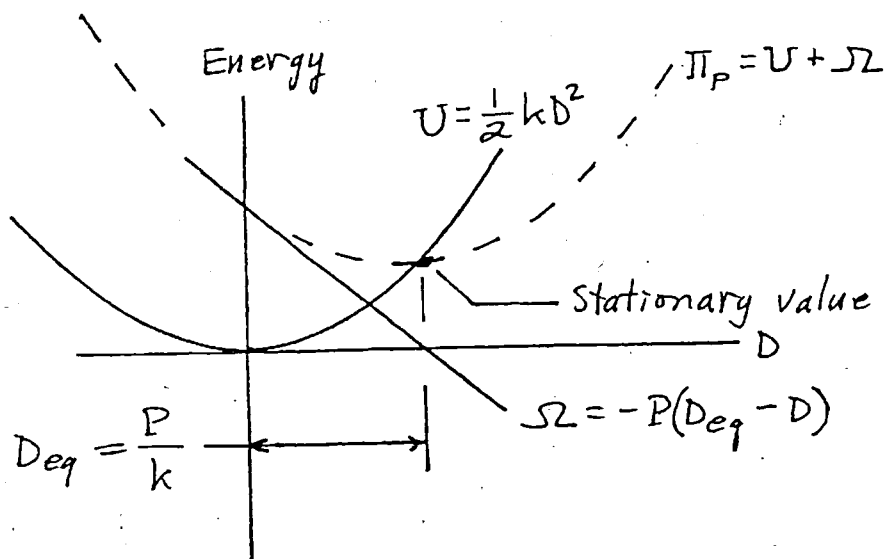
Consider the displacements of load P shown.



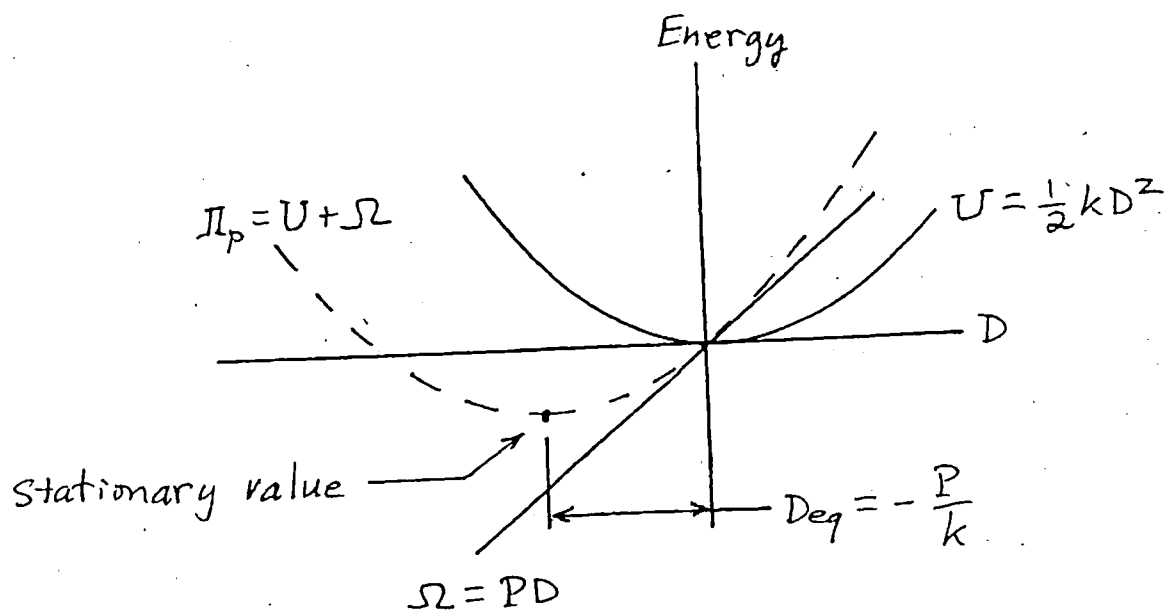
- (a) Original state, then
- (b) Translation, then
- (c) Rotation, then
- (d) Translation, then
- (e) Rotation (original state)

Only the passage from (c) to (d) involves work, here done by the horizontal component of P . Nonzero net work is done when the original state is restored. Hence, not conservative.

4.2-4



4.2-5

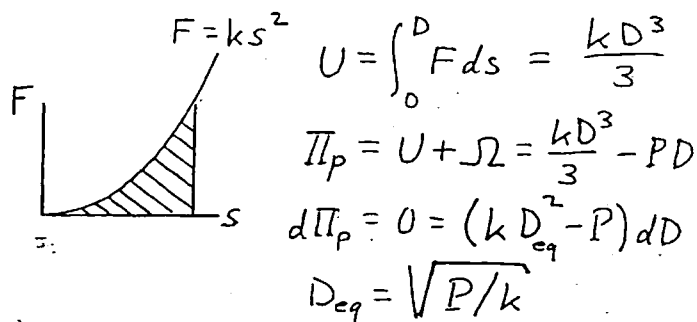


$$\Pi_p = \frac{1}{2}kD^2 + PD$$

$$d\Pi_p = (kD_{eq} + P)dD = 0$$

$$D_{eq} = -\frac{P}{k}$$

4.2-6



(Checks $F = ks^2$ for $F = P$
and $s = D_{eq}$.)

4.3-1

Let overbars denote relative displacements. Thus $D_1 = \bar{D}_1$,

$$D_2 = D_1 + \bar{D}_2, \quad D_3 = D_2 + \bar{D}_3$$

$$D_3 = D_1 + \bar{D}_2 + \bar{D}_3$$

$$\Pi_P = \frac{1}{2} k_1 D_1^2 + \frac{1}{2} k_2 \bar{D}_2^2 + \frac{1}{2} k_3 \bar{D}_3^2$$

$$- P_1 D_1 - P_2 (D_1 + \bar{D}_2) - P_3 (D_1 + \bar{D}_2 + \bar{D}_3)$$

$$\left. \begin{aligned} \frac{\partial \Pi_P}{\partial D_1} = 0 &= k D_1 - 3P, \quad D_1 = \frac{3P}{k} \\ \frac{\partial \Pi_P}{\partial \bar{D}_2} = 0 &= k \bar{D}_2 - 2P, \quad \bar{D}_2 = \frac{2P}{k} \\ \frac{\partial \Pi_P}{\partial \bar{D}_3} = 0 &= k \bar{D}_3 - P, \quad \bar{D}_3 = \frac{P}{k} \end{aligned} \right\} \begin{aligned} &\text{For } k_1 = \\ &k_2 = k_3 = k \\ &\text{and } P_1 = \\ &P_2 = P_3 = P \end{aligned}$$

$$D_2 = \frac{3P}{k} + \frac{2P}{k} = \frac{5P}{k}, \quad D_3 = \frac{5P}{k} + \frac{P}{k} = \frac{6P}{k}$$

$$\text{Eq. 4.3-5: } k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \frac{P}{k} \begin{Bmatrix} 3 \\ 5 \\ 6 \end{Bmatrix} = P \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \checkmark$$

4.3-2

Rows of $[k]$ given by $\partial U / \partial d_i = 0$.

$$U = \frac{1}{2} k e^2 = \frac{1}{2} k [d_3 - (c d_1 + s d_2)]^2$$

where $c = \cos \beta$ & $s = \sin \beta$.

$$\begin{aligned} \frac{\partial U}{\partial d_1} &= k e (-c) \\ \frac{\partial U}{\partial d_2} &= k e (-s) \\ \frac{\partial U}{\partial d_3} &= k e \\ \frac{\partial U}{\partial d_4} &= 0 \end{aligned} \quad [k] = k \begin{bmatrix} c^2 & cs & -c & 0 \\ cs & s^2 & -s & 0 \\ -c & -s & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

4.3-3 Part (a)

(a) of Problem 2.2-2

$$\Pi_P = \frac{1}{2} [k_1 u_1^2 + k_2 u_3^2 + k_3 (u_3 - u_1)^2 + k_4 (u_2 - u_1)^2] - F_1 u_1 - F_2 u_2 - F_3 u_3$$

$$\frac{\partial \Pi_P}{\partial u_1} = 0 = k_1 u_1 - k_3 (u_3 - u_1) - k_4 (u_2 - u_1) - F_1$$

$$\frac{\partial \Pi_P}{\partial u_2} = 0 = k_4 (u_2 - u_1) - F_2$$

$$\frac{\partial \Pi_P}{\partial u_3} = 0 = k_2 u_3 + k_3 (u_3 - u_1) - F_3$$

$$\begin{bmatrix} k_1 + k_3 + k_4 & -k_4 & -k_3 \\ -k_4 & k_4 & 0 \\ -k_3 & 0 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

(b) of Problem 2.2-2

$$\Pi_P = \frac{1}{2} [k_1 (u_4 - u_2)^2 + k_2 (u_2 - u_1)^2 + k_3 (u_3 - u_2)^2 + k_4 (u_4 - u_3)^2 + k_5 u_1^2 + k_6 (u_3 - u_1)^2] - F_1 u_1 - F_2 u_2 - F_3 u_3 - F_4 u_4$$

$$\frac{\partial \Pi_P}{\partial u_1} = 0 = -k_2 (u_2 - u_1) + k_5 u_1 - k_6 (u_3 - u_1) - F_1$$

$$\frac{\partial \Pi_P}{\partial u_2} = 0 = k_1 (u_4 - u_2) + k_2 (u_2 - u_1) - k_3 (u_3 - u_2) - F_2$$

$$\frac{\partial \Pi_P}{\partial u_3} = 0 = k_3 (u_3 - u_2) - k_4 (u_4 - u_3) + k_6 (u_3 - u_1) - F_3$$

$$\frac{\partial \Pi_P}{\partial u_4} = 0 = -k_1 (u_4 - u_2) + k_4 (u_4 - u_3) - F_4$$

$$\begin{bmatrix} k_5 + k_6 & -k_2 & -k_6 & 0 \\ -k_2 & k_1 + k_2 + k_3 & -k_3 & -k_1 \\ -k_6 & -k_3 & k_3 + k_4 + k_6 & -k_4 \\ 0 & -k_1 & -k_4 & k_1 + k_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

(continues)

4.3-3 (continued) Part (b)

(a) of Problem 2.2-3

$$\Pi_P = \frac{1}{2} [k_1 v_1^2 + k_2 v_2^2] \quad \frac{\partial \Pi_P}{\partial v_1} = 0 = k_1 v_1 \quad \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

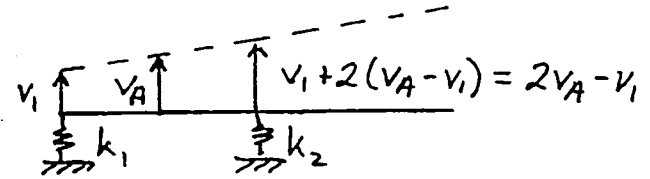
$$\frac{\partial \Pi_P}{\partial v_2} = 0 = k_2 v_2$$

(b) of Problem 2.2-3

$$\Pi_P = \frac{1}{2} [k_1 v_1^2 + k_2 (2v_A - v_1)^2]$$

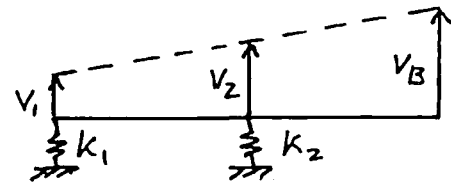
$$\frac{\partial \Pi_P}{\partial v_1} = 0 = k_1 - k_2 (2v_A - v_1)$$

$$\frac{\partial \Pi_P}{\partial v_A} = 0 = 2k_2 (2v_A - v_1)$$



$$\begin{bmatrix} k_1 + k_2 & -2k_2 \\ -2k_2 & 4k_2 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_A \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

(c) of Problem 2.2-3



$$v_1 = v_2 - (v_B - v_2) = 2v_2 - v_B$$

$$\Pi_P = \frac{1}{2} [k_1 (2v_2 - v_B)^2 + k_2 v_2^2]$$

$$\frac{\partial \Pi_P}{\partial v_2} = 0 = 2k_1 (2v_2 - v_B) + k_2 v_2$$

$$\frac{\partial \Pi_P}{\partial v_B} = -k_1 (2v_2 - v_B)$$

$$\begin{bmatrix} 4k_1 + k_2 & -2k_1 \\ -2k_1 & k_1 \end{bmatrix} \begin{Bmatrix} v_2 \\ v_B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

(continues)

4.3-3 (continued) Part (c)

(a) of Problem 2.3-2

$$\Pi_P = \frac{k}{2} \left[(u_2 - b\theta_2) - (u_1 - b\theta_1) \right]^2 - F_1 u_1 - M_1 \theta_1 - F_2 u_2 - M_2 \theta_2$$

$$\frac{\partial \Pi_P}{\partial u_1} = 0 = -k[---] - F_1$$

$$\frac{\partial \Pi_P}{\partial \theta_1} = 0 = kb[---] - M_1$$

$$\frac{\partial \Pi_P}{\partial u_2} = 0 = k[---] - F_2$$

$$\frac{\partial \Pi_P}{\partial \theta_2} = 0 = -kb[---] - M_2$$

$$\begin{bmatrix} k & -kb & -k & kb \\ -kb & kb^2 & kb & -kb^2 \\ -k & kb & k & -kb \\ kb & -kb^2 & -kb & kb^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix}$$

(b) of Problem 2.3-2

$$\Pi_P = \frac{k}{2} \langle u_1 - (u_2 - b\theta_2) \rangle^2 + \frac{k}{2} [u_2 - (u_1 + b\theta_1)]^2 - F_1 u_1 - M_1 \theta_1 - F_2 u_2 - M_2 \theta_2$$

$$\frac{\partial \Pi_P}{\partial u_1} = 0 = k \langle --- \rangle - k[---] - F_1 = k(u_1 - u_2 + b\theta_2 - u_2 + u_1 + b\theta_1) - F_1$$

$$\frac{\partial \Pi_P}{\partial \theta_1} = 0 = -kb[---] - M_1 = kb(u_1 - u_2 + b\theta_1) - M_1$$

$$\frac{\partial \Pi_P}{\partial u_2} = 0 = -k \langle --- \rangle + k[---] - F_2 = k(-u_1 + u_2 - b\theta_2 + u_2 - u_1 - b\theta_1) - F_2$$

$$\frac{\partial \Pi_P}{\partial \theta_2} = 0 = kb \langle --- \rangle - M_2 = kb(u_1 - u_2 + b\theta_2) - M_2$$

$$\begin{bmatrix} 2k & kb & -2k & kb \\ kb & kb^2 & -kb & 0 \\ -2k & -kb & 2k & -kb \\ kb & 0 & -kb & kb^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix}$$

(continues)

4.3-3 (concluded) Part (d)

(a) of Problem 2.3-3

$$\Pi_P = \frac{1}{2} k_A u_A^2 + \frac{1}{2} k_B \left(\frac{u_C - u_A}{4L} \right)^2$$

$$\frac{\partial \Pi_P}{\partial u_A} = 0 = k_A u_A - \frac{k_B}{4L} \left(\frac{u_C - u_A}{4L} \right)$$

$$\frac{\partial \Pi_P}{\partial u_C} = 0 = \frac{k_B}{4L} \left(\frac{u_C - u_A}{4L} \right)$$

$$\begin{bmatrix} k_A + \frac{k_B}{16L^2} & -\frac{k_B}{16L^2} \\ -\frac{k_B}{16L^2} & \frac{k_B}{16L^2} \end{bmatrix} \begin{Bmatrix} u_A \\ u_C \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

(b) of Problem 2.3-3

$$\Pi_P = \frac{1}{2} k (u - \theta b)^2 + \frac{1}{2} k (2a\theta)^2 - Fu - M\theta$$

$$\frac{\partial \Pi_P}{\partial u} = 0 = k(u - \theta b) - F$$

$$\frac{\partial \Pi_P}{\partial \theta} = 0 = -kb(u - \theta b) + k(2a\theta)(2a) - M$$

$$\begin{bmatrix} k & -kb \\ -kb & kb^2 + 4ka^2 \end{bmatrix} \begin{Bmatrix} u \\ \theta \end{Bmatrix} = \begin{Bmatrix} F \\ M \end{Bmatrix}$$

4.3-4

$$[K]\{D\} = \begin{Bmatrix} k_1 D_1 + k_2 (D_1 - D_2) \\ k_2 (D_2 - D_1) + k_3 (D_2 - D_3) \\ k_3 (D_3 - D_2) \end{Bmatrix}$$

$$\frac{1}{2} \{D\}^T ([K]\{D\}) = \frac{1}{2} [k_1 D_1^2 + k_2 (D_1^2 - D_1 D_2 - D_1 D_2 + D_2^2) + k_3 (D_2^2 - D_2 D_3 - D_2 D_3 + D_3^2)]$$

$$\frac{1}{2} \{D\}^T [K]\{D\} = \frac{1}{2} k_1 D_1^2 + \frac{1}{2} k_2 (D_1^2 - 2D_1 D_2 + D_2^2) + \frac{1}{2} k_3 (D_2^2 - 2D_2 D_3 + D_3^2) \quad \checkmark$$

$$\Omega = -\{D\}^T \{R\} = -D_1 P_1 - D_2 P_2 - D_3 P_3$$

4.4-1

Uniaxial stress: $\{\underline{\epsilon}\} = \epsilon_x [1 \ -\nu \ 0]^T$.

Then

$$[\underline{E}]\{\underline{\epsilon}\} = \frac{E\epsilon_x}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} 1 \\ -\nu \\ 0 \end{Bmatrix}$$

$$\frac{1}{2} \{\underline{\epsilon}\}^T ([\underline{E}]\{\underline{\epsilon}\}) = \frac{\epsilon_x}{2} [1 \ -\nu \ 0] \begin{Bmatrix} E\epsilon_x \\ 0 \\ 0 \end{Bmatrix} = \frac{E}{2} \epsilon_x^2$$

Checks Eq. 4.4-9.

4.4-2

Add to 1st integral the terms

$$\int [-E\epsilon_x \epsilon_{x0} + \epsilon_x \sigma_{x0}] dV, \text{ where}$$

$$\epsilon_{x0} = -z\kappa_0, \sigma_{x0} = -m_0 z/I, dV = dA dx,$$

and $\epsilon_x = -z w_{,xx}$. Thus

$$\iiint [-E z^2 \kappa_0 w_{,xx} - z w_{,xx} \left(-\frac{m_0 z}{I}\right)] dA dx =$$

$$\int I [-E \kappa_0 w_{,xx} + w_{,xx} \frac{m_0}{I}] dx = \int w_{,xx} [-EI \kappa_0 + m_0] dx$$

4.4-3

Now $\sigma_0 = 0$, $\epsilon_0 = \alpha T$, so Eq. 4.4-12 becomes

$$\Pi_P = \int_0^L \left[\frac{1}{2} E \left(\frac{D}{L} \right)^2 - \frac{D}{L} E (\alpha T) \right] A dx - DP = \frac{EAD^2}{2L} - DE\alpha T - DP$$

Same as latter form of Eq. 4.4-12, so same result is obtained.

4.5-1

4.5-2

Set up using 3 terms, then drop a_3 to do Problem 4.5-1.

$$u = a_1 x + a_2 x^2 + a_3 x^3$$

$$u_{,x} = a_1 + 2a_2 x + 3a_3 x^2$$

$$\int_0^{L_T} u_{,x}^2 dx = a_1^2 L_T + \frac{4}{3} a_2^2 L_T^3 + \frac{9}{5} a_3^2 L_T^5 + 2a_1 a_2 L_T^2 + 2a_1 a_3 L_T^3 + 3a_2 a_3 L_T^4$$

$$\int_0^{L_T} c x u dx = c \left(\frac{1}{3} a_1 L_T^3 + \frac{1}{4} a_2 L_T^4 + \frac{1}{5} a_3 L_T^5 \right)$$

$$\Pi_P = \frac{AE}{2} (1^{st} \text{ integral}) - (2^{nd} \text{ integral})$$

4.5-1

Set $a_3 = 0$ to do Problem 4.5-1. Then

$$0 = \frac{\partial \Pi_P}{\partial a_1} = AE (a_1 L_T + a_2 L_T^2) - \frac{1}{3} c L_T^3$$

$$0 = \frac{\partial \Pi_P}{\partial a_2} = AE \left(\frac{4}{3} a_2 L_T^3 + a_1 L_T^2 \right) - \frac{1}{4} c L_T^4$$

Checks the first of Eqs. 4.5-6a.

4.5-2

Now retain the a_3 term in Π_P .

$$\frac{\partial \Pi_P}{\partial a_1} = 0 = AE (a_1 L_T + a_2 L_T^2 + a_3 L_T^3) - \frac{1}{3} c L_T^3$$

$$\frac{\partial \Pi_P}{\partial a_2} = 0 = AE \left(\frac{4}{3} a_2 L_T^3 + a_1 L_T^2 + \frac{3}{2} a_3 L_T^4 \right) - \frac{c L_T^4}{4}$$

$$\frac{\partial \Pi_P}{\partial a_3} = 0 = AE \left(\frac{9}{5} a_3 L_T^5 + a_1 L_T^3 + \frac{3}{2} a_2 L_T^4 \right) - \frac{c L_T^5}{5}$$

In Matrix format,

$$A = \begin{bmatrix} 1 & L_T & L_T^2 \\ L_T & \frac{4}{3} L_T^2 & \frac{3}{2} L_T^3 \\ L_T^2 & \frac{3}{2} L_T^3 & \frac{9}{5} L_T^4 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{c L_T^2}{60} \begin{Bmatrix} 20 \\ 15 L_T \\ 12 L_T^2 \end{Bmatrix}$$

Satisfied by Eqs. 4.5-9.

4.5-3

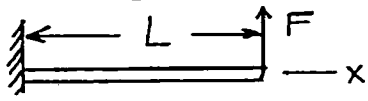
Differential equation, Eq. 4.5-7: $AEu_{,xx} + cx = 0$

Eq. 4.5-5c: $u = \frac{cL_T^2}{3AE}x$, $AE(0) + cx = 0$
Satisfied only at $x = 0$

Eq. 4.5-6b: $u = \frac{cL_T}{12AE}(7L_Tx - 3x^2)$, $AE\left[\frac{cL_T}{12AE}(-6)\right] + cx = 0$
Satisfied only at $x = \frac{L_T}{2}$

Eq. 4.5-8: $u = \frac{c}{6AE}(3L_T^2x - x^3)$, $AE\left[\frac{c}{6AE}(-6x)\right] + cx = 0$
Satisfied for all x .

4.5-4



(a) $v = a_1 x^3, v_{,x} = 3a_1 x^2$

Admissible, as $v = v_{,x} = 0$ @ $x = 0$.

But poor, as it omits lowest-order admissible term, namely x^2 .

(b) $v = a_1 x^2 + a_2 x^3 + a_3 x^4$

(c) Will be exact, since exact v is cubic in x for this load. $\therefore a_3 = 0$.

(d) $\Pi_p = \int_0^L \frac{EI}{2} (6a_1 x)^2 dx - F(a_1 L^3)$

$\frac{d\Pi_p}{da_1} = \frac{d}{da_1} (6EIL^3 a_1^2 - FL^3 a_1) = 0; a_1 = F/12EI$
 At $x = L$, gives $v = FL^3/12EI$

4.5-5

Assume that uniform q acts up.

(a) $v = a_1 x (L-x)$ admissible

$$\Pi_P = \frac{EI}{2} \int_0^L (2a_1)^2 dx - \int_0^L q a_1 x (L-x) dx$$

$$\frac{d\Pi_P}{da_1} = 0 = \frac{d}{da_1} \left(2EILa_1^2 - \frac{qL^3}{6} a_1 \right), \quad a_1 = \frac{qL^2}{24EI}$$

At center, $x = L/2$,

$$v = a_1 (L^2/4) = 0.010417 (qL^4/EI)$$

$$M = EI v_{,xx} = -0.08333 qL^2$$

(b) $v = a_1 \sin \frac{\pi x}{L}$, $v_{,xx} = -a_1 \frac{\pi^2}{L^2} \sin \frac{\pi x}{L}$

Substitute $\theta = \frac{\pi x}{L}$, $dx = \frac{L}{\pi} d\theta$

$$\Pi_P = \frac{EI}{2} \frac{a_1^2 \pi^4}{L^4} \frac{L}{\pi} \int_0^\pi \sin^2 \theta d\theta - q a_1 \frac{L}{\pi} \int_0^\pi \sin \theta d\theta$$

$$\Pi_P = \frac{EI \pi^4}{4L^3} a_1^2 - \frac{2qL}{\pi} a_1$$

$$d\Pi_P/da_1 = 0 \text{ yields } a_1 = \frac{4qL^4}{\pi^5 EI}$$

At center, $x = L/2$,

$$v = \frac{4qL^4}{\pi^5 EI} = 0.013071 (qL^4/EI)$$

$$M = EI v_{,xx} = EI \left(-\frac{\pi^2 a_1}{L^2} \right) = -0.12901 qL^2$$

(c) Exact values at center:

$$v = \frac{5qL^4}{384EI} = 0.013021 \frac{qL^4}{EI}, \quad M = -\frac{qL^2}{8}$$

Assumption (a) satisfies only essential BC's; assumption (b) also satisfies non-essential BC's, so of course works better.

4.5-6

Exact answers are

$$v_L = \frac{P_L L^3}{3EI} - \frac{M_L L^2}{2EI} - \frac{q L^4}{8EI}$$

$$(v, x)_L = \frac{P_L L^2}{2EI} - \frac{M_L L}{EI} - \frac{q L^3}{6EI}$$

Set-up for (a) and (b):

$$v = a_1 x^2 + a_2 x^3, \quad v, x = 2a_1 x + 3a_2 x^2$$

$$v, xx = 2a_1 + 6a_2 x, \quad \Pi_P = U + \Omega$$

$$U = \int_0^L \frac{EI}{2} v, xx^2 dx = 2EIL(a_1^2 + 3a_1 a_2 L + 3a_2^2 L^2)$$

$$\Omega = -P_L v_L + M_L (v, x)_L + \int_0^L q v dx$$

$$\Omega = -P_L (a_1 L^2 + a_2 L^3) + M_L (2a_1 L + 3a_2 L^2) + q \left(\frac{a_1 L^3}{3} + \frac{a_2 L^4}{4} \right)$$

$$\frac{\partial \Pi_P}{\partial a_1} = 0 = 2EIL(2a_1 + 3a_2 L) - P_L L^2 + M_L (2L) + \frac{q L^3}{3}$$

$$\frac{\partial \Pi_P}{\partial a_2} = 0 = 2EIL(3a_1 L + 6a_2 L^2) - P_L L^3 + M_L (3L^2) + \frac{q L^4}{4}$$

(a) Set $a_2 = 0$; from 1st eq.

$$a_1 = \frac{P_L L}{4EI} - \frac{M_L}{2EI} - \frac{q L}{12EI}$$

$$v_L = a_1 L^2 = \frac{P_L L^3}{4EI} - \frac{M_L L^2}{2EI} - \frac{q L^4}{12EI}$$

$$(v, x)_L = 2a_1 L = \frac{P_L L^2}{2EI} - \frac{M_L L}{EI} - \frac{q L^3}{6EI}$$

$$(b) \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{1}{6EIL^2} \begin{Bmatrix} 3P_L L^3 - 3M_L L^2 + \frac{5}{4}q L^4 \\ -P_L L^2 + \frac{1}{2}q L^3 \end{Bmatrix}$$

$$v_L = a_1 L^2 + a_2 L^3 = \frac{P_L L^3}{3EI} - \frac{M_L L^2}{2EI} - \frac{q L^4}{8EI}$$

$$(v, x)_L = 2a_1 L + 3a_2 L^2 = \frac{P_L L^2}{2EI} - \frac{M_L L}{EI} - \frac{q L^3}{6EI}$$

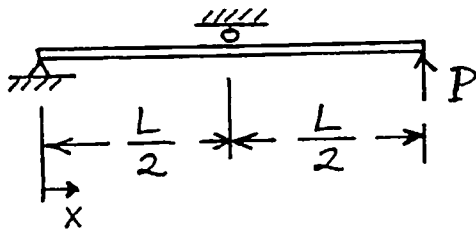
(c) Only when $P_L = q = 0$, since

$v = a_1 x^2$ only for M_L loading.

4.5-7

An infinite number. We are trying to model, by one polynomial series, two different polynomials that meet with the same v & same v_x at $x = \frac{L}{2}$.

4.5-8



$$v = a_0 + a_1 x + a_2 x^2$$

$$v = 0 \text{ at } x = 0; \quad a_0 = 0$$

$$v = 0 \text{ at } x = \frac{L}{2}; \quad a_1 \frac{L}{2} + a_2 \frac{L^2}{4}; \quad a_2 = -\frac{2}{L} a_1$$

$$\therefore v = a_1 \left(x - \frac{2}{L} x^2 \right)$$

$$v_{,x} = a_1 \left(1 - \frac{4}{L} x \right)$$

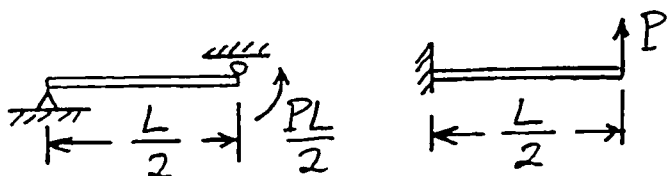
$$v_{,xx} = -a_1 \frac{4}{L}$$

$$\Pi_P = \int_0^L \frac{EI}{2} \left(-a_1 \frac{4}{L} \right)^2 dx - a_1 \left(L - \frac{2}{L} L^2 \right) = \frac{EI}{2} a_1^2 \frac{16}{L^2} L + a_1 LP$$

$$\frac{d\Pi_P}{da_1} = 0 = \frac{16EI}{L} a_1 + PL, \quad a_1 = -\frac{PL^2}{16EI}$$

$$\text{At } x=L, \quad v = -a_1 L = \frac{PL^3}{16EI}$$

Elementary beam theory:



$$v = \Delta \frac{L}{2} + \frac{P(L/2)^3}{3EI}$$

$$v = \frac{(PL/2)(L/2)}{3EI} \frac{L}{2} + \frac{PL^3}{24EI} = \frac{PL^3}{3EI} \left(\frac{1}{8} + \frac{1}{8} \right) = \frac{PL^3}{12EI}$$

$$\frac{1/12}{1/12} 100\% = \left(\frac{3}{4} - 1 \right) 100\% = -25\%$$

$$M = EI v_{,xx} = EI \left(-\frac{4}{L} \right) \left(-\frac{PL^2}{16EI} \right) = \frac{PL}{4}$$

(for all x)

error is -50%
at $x = \frac{L}{2}$

4.5-9

$$\left. \begin{aligned} v &= a_0 + a_1 x + a_2 x^2, \quad v_{,x} = a_1 + 2a_2 x \\ v &= 0 \text{ at } x=0; \quad a_0 = 0 \\ v_{,x} &= 0 \text{ at } x=L; \quad a_1 = -2a_2 L \end{aligned} \right\} \begin{aligned} v &= a_2 (-2Lx + x^2) \\ v_{,x} &= a_2 (-2L + 2x) \\ v_{,xx} &= 2a_2 \end{aligned}$$

$$\Pi_P = \int_0^L \frac{EI}{2} (v_{,xx})^2 dx + P v_L = \frac{EIL}{2} (4a_2^2) + P(-L^2)a_2$$

$$\frac{d\Pi_P}{da_2} = 0 = 4EILa_2 - PL^2; \quad a_2 = \frac{PL}{4EI}$$

$$\text{At } x=L, \quad v = -L^2 a_2 = -\frac{PL^3}{4EI}$$

$$\text{Beam theory: at } x=L, \quad v = -\frac{PL^3}{3EI}$$

$$\frac{1/4 - 1/3}{1/3} 100\% = \left(\frac{3}{4} - 1\right) 100\% = -25\%$$

$$M = EI v_{,xx} = EI(2a_2) = \frac{PL}{2} \\ (\text{for all } x)$$

error is -50%
at $x=L$

4.5-10

$$u = a_0 + a_1 x + a_2 x^2$$

$$\left. \begin{array}{l} u=0 \text{ at } x=0; \quad a_0=0 \\ u=0 \text{ at } x=L; \quad a_2 = -\frac{a_1}{L} \end{array} \right\} \begin{array}{l} u = a_1 \left(x - \frac{x^2}{L} \right) \\ u_{,x} = a_1 \left(1 - \frac{2x}{L} \right) \end{array}$$

Eq. 4.4-9:

$$\Pi_p = \int_0^L \left(\frac{1}{2} E u_{,x}^2 - u_{,x} E \epsilon_{x0} \right) A dx \quad \text{where } \epsilon_{x0} = \alpha \frac{T_0 x}{L}$$

$$\Pi_p = \int_0^L \left[\frac{E}{2} a_1^2 \left(1 - \frac{4x}{L} + \frac{4x^2}{L^2} \right) - \frac{E \alpha T_0}{L} a_1 \left(x - \frac{2x^2}{L} \right) \right] A dx$$

$$\frac{d\Pi_p}{da_1} = 0, \quad 0 = \left[a_1 \left(x - \frac{2x^2}{L} + \frac{4x^3}{3L^2} \right) - \frac{\alpha T_0}{L} \left(\frac{x^2}{L} - \frac{2x^3}{3L} \right) \right]_0^L$$

$$0 = a_1 \frac{L}{3} - \frac{\alpha T_0}{L} \left(-\frac{L^2}{6} \right), \quad a_1 = -\frac{\alpha T_0}{2}$$

$$u = -\frac{\alpha T_0}{2} \left(x - \frac{x^2}{L} \right)$$

Stress field:

$$\sigma = E \epsilon_x + \sigma_0 = E u_{,x} + \left(-E \alpha \frac{T_0 x}{L} \right)$$

$$\sigma = E \left[-\frac{\alpha T_0}{2} \left(1 - \frac{2x}{L} \right) - \frac{\alpha T_0 x}{L} \right]$$

$$\sigma = -\frac{E \alpha T_0}{2} \quad \checkmark$$

4.5-11

$$u = a_1 x$$

$$u_{,x} = a_1$$

$$v = b_1 x^2$$

$$v_{,x} = 2b_1 x$$

$$v_{,xx} = 2b_1$$

$$\begin{aligned}\Pi_P &= \int_0^L \frac{AE}{L} a_1^2 dx + \int_0^L \frac{P}{2} 4b_1^2 x^2 dx + \int_0^L \frac{EI}{2} 4b_1^2 dx - P(a_1 L) \\ &= \frac{AEL}{2} a_1^2 + \frac{2PL^3}{3} b_1^2 + 2EIL b_1^2 - PL a_1\end{aligned}$$

$$\frac{\partial \Pi_P}{\partial a_1} = 0 = AEL a_1 - PL \quad \longrightarrow \quad a_1 = \frac{P}{AE}, \quad \therefore \sigma_x = E u_{,x} = \frac{P}{A} \quad \checkmark$$

$$\frac{\partial \Pi_P}{\partial b_1} = 0 = \frac{4PL^3}{3} b_1 + 4EIL b_1$$

$$0 = b_1 \left(\frac{4PL^3}{3} + 4EIL \right)$$

must vanish if $b_1 \neq 0 \longrightarrow P = -\frac{3EI}{L^2}$

Exact P_{cr} is $P_{cr} = -\frac{\pi^2 EI}{4L^2} = -2.467 \frac{EI}{L^2}$

4.5-12

$$v = a_1 x(L-x)$$

$$q = q_0 \left(1 - \frac{x}{L}\right)$$

$$v_{L/2} = \frac{a_1 L^2}{4}$$

$$v_{,x} = a_1 (L-2x)$$

$$(v_{,x})_L = -La_1$$

$$v_{,xx} = -2a_1$$

$$\Pi_p = \int_0^L \frac{EI}{2} v_{,xx}^2 dx + \int_0^L q v dx + \frac{1}{2} k v_{L/2}^2 - M_L (v_{,x})_L$$

$$\Pi_p = \int_0^L \frac{EI}{2} 4a_1^2 dx + \int_0^L q_0 a_1 \left(Lx - 2x^2 + \frac{x^3}{L}\right) dx + \frac{1}{2} k \frac{a_1^2 L^4}{16} - M_L (-La_1)$$

$$\Pi_p = 2EILa_1^2 + q_0 a_1 \left(\frac{Lx^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4L}\right)_0^L + \frac{kL^4}{32} a_1^2 + M_L La_1$$

$$\Pi_p = 2EILa_1^2 + \frac{q_0 L^3}{12} a_1 + \frac{kL^4}{32} a_1^2 + M_L La_1$$

$$\frac{d\Pi_p}{da_1} = 0 = 4EILa_1 + \frac{q_0 L^3}{12} + \frac{kL^4}{16} a_1 + M_L L$$

$$a_1 = - \frac{\frac{q_0 L^3}{12} + M_L L}{4EIL + \frac{kL^4}{12}} = - \frac{\frac{q_0}{L} + \frac{12M_L}{L^3}}{\frac{48EI}{L^3} + k}$$

$$v = a_1 x(L-x) = - \frac{\frac{q_0}{L} + \frac{12M_L}{L^3}}{\frac{48EI}{L^3} + k} x(L-x)$$

4.5-13

⊄ symm: need i odd only.

$$v = \sum a_i \sin \frac{i\pi x}{L}, \quad v_{xx} = \sum -a_i \left(\frac{i\pi}{L}\right)^2 \sin \frac{i\pi x}{L}$$

$$U = \sum \frac{EI}{2} a_i^2 \left(\frac{i\pi}{L}\right)^4 \frac{L}{\pi} \int_0^\pi \sin^2 \theta d\theta = \sum \frac{EI i^4 \pi^4}{4L^3} a_i^2$$

$$\Omega = +P v_{L/2} = P \sum a_i \sin \frac{i\pi}{2}, \quad \Pi_P = U + \Omega$$

$$\frac{d\Pi_P}{da_i} = 0 = \frac{EI i^4 \pi^4}{2L^3} a_i + P \sin \frac{i\pi}{2}$$

$$a_i = -\frac{2PL^3}{EI\pi^4} \sum \frac{1}{i^4} \sin \frac{i\pi}{2}, \quad v_{L/2} = -\frac{2PL^3}{EI\pi^4} \sum \frac{1}{i^4}$$

$$M = EI v_{xx} = EI \frac{2PL^3}{EI\pi^4} \sum \left(\frac{i\pi}{L}\right)^2 \frac{1}{i^4} \sin^2 \frac{i\pi}{2}$$

$$M_{L/2} = \frac{2PL}{\pi^2} \sum \frac{1}{i^2}. \quad \text{At center, } x=L/2,$$

$$v_{L/2} = -C_D \frac{PL^3}{EI} \quad \& \quad M_{L/2} = C_M PL, \quad \text{where}$$

	<u>$n=1$</u>	<u>$n=2$</u>	<u>$n=3$</u>	<u>$n=4$</u>	<u>exact</u>
C_D	.020532	.020785	.020818	.020827	.020833
C_M	.20264	.22516	.23326	.23740	.25000

4.5-14

⊘ symm: need i odd only.

U is the same as in Problem 4.5-13.

$$\Omega = \int_0^L q v dx = q \sum a_i \int_0^L \sin \frac{i\pi x}{L} dx = a_i \frac{2qL}{\pi} \sum \frac{1}{i}$$

$$\frac{d\Pi_p}{da_i} = 0 = \frac{EI i^4 \pi^4}{2L^3} a_i + \frac{2qL}{i\pi}, a_i = -\frac{4qL^4}{EI \pi^5 i^5}$$

At the center, $x=L/2$,

$$v_{\frac{L}{2}} = -\frac{4qL^4}{EI \pi^5} \left(1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \dots \right)$$

$$M_{\frac{L}{2}} = (EI v_{xx})_{\frac{L}{2}} = \frac{4qL^2}{\pi^3} \left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots \right)$$

$$\text{Let } v_{\frac{L}{2}} = -C_D \frac{qL^4}{EI} \quad \& \quad M_{\frac{L}{2}} = C_M (qL^2)$$

$$\begin{array}{ccccc} n=1 & n=2 & n=3 & n=4 & \text{exact} \end{array}$$

$$C_D \quad .013071 \quad .013017 \quad .013021 \quad .013021 \quad .013021$$

$$C_M \quad .12901 \quad .12423 \quad .12526 \quad .12488 \quad .12500$$

4.6-1

$$(a) W_e = \int_0^{L_T} q u dx = \frac{c^2}{6AE} \int_0^{L_T} (3L_T^2 x^2 - x^4) dx$$

$$W_e = \frac{c^2}{6AE} \left(L_T^2 x^3 - \frac{x^5}{5} \right)_0^{L_T} = 0.13333 \frac{c^2 L_T^5}{AE}$$

(b) Eq. 4.5-5

$$W_1 = \int_0^{L_T} q u dx = \frac{c^2 L_T^2}{3AE} \int_0^{L_T} x^2 dx = 0.11111 \frac{c^2 L_T^5}{AE}$$

Eq. 4.5-6

$$W_2 = \int_0^{L_T} q u dx = \frac{c^2 L_T}{12AE} \int_0^{L_T} (7L_T x^2 - 3x^3) dx$$

$$W_2 = 0.13194 (c^2 L_T^5 / AE)$$

$W_e > W_2 > W_1$. Displacement u is underestimated (in an integral sense) but gets more exact as more terms used.

4.6-2

$$(a) \epsilon_x = \frac{c}{6AE} (3L_T^2 - 3x^2)$$

$$U_e = \int_0^{L_T} \frac{1}{2} E \epsilon_x^2 A dx = \frac{EA c^2}{8A^2 E^2} \int_0^{L_T} (L_T^2 - x^2)^2 dx$$

$$U_e = 0.06667 \frac{c^2 L_T^5}{AE} = \frac{1}{2} W_e \text{ from 3.26(a)}$$

As expected: work done by gradually applied load is $\frac{1}{2} W_e$ & is stored as strain energy

$$(b) U_1 = \int_0^{L_T} \frac{1}{2} E \epsilon_x^2 A dx = \frac{EA}{2} \left(\frac{c L_T^2}{3AE} \right)^2 \int_0^{L_T} dx$$

$$U_1 = 0.05556 \frac{c^2 L_T^5}{AE} = \frac{1}{2} W_1 \text{ from 3.26(b)}$$

$$U_2 = \int_0^{L_T} \frac{1}{2} E \epsilon_x^2 A dx = \frac{EA}{2} \left(\frac{c L_T}{12AE} \right)^2 \int_0^{L_T} (7L_T - 6x)^2 dx$$

$$U_2 = 0.065972 \frac{c^2 L_T^5}{AE} = \frac{1}{2} W_2 \text{ from 3.26(b)}$$

4.7-1

(a) Apply Eq. 4.7-2a: $\frac{\partial F}{\partial \phi} - \frac{d}{dx} \frac{\partial F}{\partial \phi_x} + \frac{d^2}{dx^2} \frac{\partial F}{\partial \phi_{xx}} = 0$

$$(2c_3\phi + c_4) - \frac{d}{dx}(2c_2\phi_x) + \frac{d^2}{dx^2}(2c_1\phi_{xx}) = 0$$

$$2c_1\phi_{xxxx} - 2c_2\phi_{xx} + 2c_3\phi + c_4 = 0$$

(b) $\Pi = \int [c_1\phi_{xx}^2 + c_2\phi_x^2 + c_3\phi^2 + c_4\phi + c_5] dx$

Let $\phi + \delta\phi = \phi + e\eta$

$$\Pi + \delta\Pi = \int [c_1(\phi_{xx} + e\eta_{xx})^2 + c_2(\phi_x + e\eta_x)^2 + c_3(\phi + e\eta)^2 + c_4(\phi + e\eta) + c_5] dx$$

$$\int (\phi_{xx} + e\eta_{xx})^2 dx = \int (\phi_{xx}^2 + 2e\phi_{xx}\eta_{xx} + e^2\eta_{xx}^2) dx$$

$\nwarrow$ discard

$$= \int (\phi_{xx}^2 + 2e\phi_{xx}\eta_{xx}) dx$$

$$= \int \phi_{xx}^2 dx - 2e \int \phi_{xxx}\eta_x dx + \text{B.C. term}$$

$$= \int \phi_{xx}^2 dx + 2e \int \phi_{xxxx}\eta dx + \text{B.C. terms}$$

$$\int (\phi_x + e\eta_x)^2 dx = \int (\phi_x^2 + 2e\phi_x\eta_x + e^2\eta_x^2) dx$$

$\nwarrow$ discard

$$= \int \phi_x^2 dx + 2e \int \phi_x\eta_x dx = \int \phi_x^2 dx - 2e \int \phi_{xx}\eta dx + \text{B.C. term}$$

$$\int (\phi + e\eta)^2 dx = \int (\phi^2 + 2e\eta\phi + e^2\eta^2) dx$$

$\nwarrow$ discard

$$\delta\Pi = (\Pi + \delta\Pi) - \Pi = e \underbrace{\int (2c_1\phi_{xxxx} - 2c_2\phi_{xx} + 2c_3\phi + c_4) \eta dx}_{\text{must vanish}} + \text{B.C. terms}$$

4.7-2

$$\text{Eq. 4.7-2a: } \frac{\partial F}{\partial \phi} - \frac{d}{dx} \frac{\partial F}{\partial \phi_x} = 0$$

$$-50 - \frac{d}{dx} \phi_x = 0 \quad \text{i.e. } \phi_{xx} + 50 = 0$$

$$\therefore \phi_x = -50x + C_1 \quad \& \quad \phi = -25x^2 + C_1x + C_2$$

$$\text{Impose B.C.'s: } \phi = 0 \text{ @ } x=0, \text{ so } C_2 = 0$$

$$20 = -25L^2 + C_1L, \text{ so } C_1 = \frac{20}{L} + 25L$$

$$\phi = -25x^2 + \left(\frac{20}{L} + 25L\right)x$$

4.7-3

Use Eq. 4.7-2a with $F = \frac{EI_z}{2} v_{,xx}^2 - qv$

$$\frac{\partial F}{\partial v} = -q \quad \text{and} \quad \frac{\partial F}{\partial v_{,xx}} = EI_z v_{,xx}$$

$$\frac{\partial F}{\partial v} + \frac{d^2}{dx^2} \frac{\partial F}{\partial v_{,xx}} \quad \text{yields} \quad (EI_z v_{,xx})_{,xx} - q = 0$$

$$\text{If } EI_z \text{ is constant, } EI_z v_{,xxxx} - q = 0$$

4.7-4

$$\Pi_P = \int_0^L \left[\frac{1}{2} E I_2 v_{,xx} - q v \right] dx \quad \text{Let } v + \delta v = v + \epsilon \eta$$

$$\begin{aligned}\Pi_p + \delta \Pi_p &= \int_0^L \left[\frac{1}{2} E I_z (v_{,xx} + e \eta_{,xx})^2 - q(v + e \eta) \right] dx \\ &= \int_0^L \left[\frac{1}{2} E I_z (v_{,xx}^2 + 2e v_{,xx} \eta_{,xx} + \underbrace{e^2 \eta_{,xx}^2}_{\text{discard}}) - q(v + e \eta) \right] dx\end{aligned}$$

Two integrations by parts:

$$\int_0^L EI_2 v_{,xx} \eta_{,xx} dx = - \int_0^L (EI_2 v_{,xx})_{,x} \eta_{,x} dx + \text{B.C. term}$$

$$= \int_0^L (EI_2 v_{,xx})_{,xx} \eta dx + \text{B.C. terms}$$

$$\Pi_P + \delta \Pi_P = \int_0^L \left[\frac{1}{2} E I_z v_{,xx}^2 + e (E I_z v_{,xx})_{,xx} \eta - q(v + e\eta) \right] dx + B.C. \text{ terms}$$

$$\delta \Pi_p = (\Pi_p + \delta \Pi_p) - \Pi_p = \int_0^L \underbrace{e \left[(E \epsilon_{xx})_{,xx} - q \right]}_{\text{must vanish}} \eta \, dx + \text{B.C. terms}$$

$$\boxed{4.7-5} \quad F = \frac{D}{2} \left[w_{,xx}^2 + 2w_{,xx}w_{,yy} + w_{,yy}^2 - 2(1-\nu)(w_{,xx}w_{,yy} - w_{,xy}^2) - \frac{2q}{D}w \right]$$

$$\frac{\partial F}{\partial w} = -q \quad \frac{\partial F}{\partial w_{,x}} = 0 \quad \frac{\partial F}{\partial w_{,y}} = 0$$

$$\frac{\partial F}{\partial w_{,xx}} = D \left[w_{,xx} + w_{,yy} - (1-\nu)w_{,yy} \right]$$

$$\frac{\partial F}{\partial w_{,yy}} = D \left[w_{,xx} + w_{,yy} - (1-\nu)w_{,xx} \right]$$

$$\frac{\partial F}{\partial w_{,xy}} = 2D(1-\nu)w_{,xy}$$

$$\frac{\partial^2}{\partial x^2} \frac{\partial F}{\partial w_{,xx}} = D \left[w_{,xxxx} + w_{,xxyy} - (1-\nu)w_{,xxyy} \right] = D \left[w_{,xxxx} + \nu w_{,xxyy} \right]$$

$$\frac{\partial^2}{\partial y^2} \frac{\partial F}{\partial w_{,yy}} = D \left[w_{,xxyy} + w_{,yyyy} - (1-\nu)w_{,xxyy} \right] = D \left[w_{,yyyy} + \nu w_{,xxyy} \right]$$

$$\frac{\partial^2}{\partial x \partial y} \frac{\partial F}{\partial w_{,xy}} = 2D(1-\nu)w_{,xxyy}$$

$$\text{Substitute into } \frac{\partial F}{\partial w} + \frac{\partial^2}{\partial x^2} \frac{\partial F}{\partial w_{,xx}} + \frac{\partial^2}{\partial y^2} \frac{\partial F}{\partial w_{,yy}} + \frac{\partial^2}{\partial x \partial y} \frac{\partial F}{\partial w_{,xy}} = 0$$

$$-q + D \left[w_{,xxxx} + \nu w_{,xxyy} + w_{,yyyy} + \nu w_{,xxyy} + 2(1-\nu)w_{,xxyy} \right] = 0$$

$$-q + D(w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy}) = 0$$

$$\therefore \nabla^4 w = 0$$

$$\nabla^4 w = \frac{q}{D}$$

4.7-6

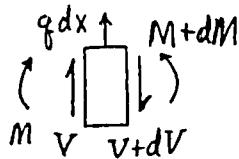
$$\frac{\partial F}{\partial M} = \frac{M}{EI}, \quad \frac{\partial F}{\partial M_{,x}} = v_{,x}$$

$$\frac{d}{dx} \frac{\partial F}{\partial M_{,x}} - \frac{\partial F}{\partial M} = v_{,xx} - \frac{M}{EI} = 0 \quad \checkmark$$

This is the moment-curvature relation.

$$\frac{\partial F}{\partial v} = q, \quad \frac{\partial F}{\partial v_{,x}} = M_{,x}$$

$$\frac{d}{dx} \frac{\partial F}{\partial v_{,x}} - \frac{\partial F}{\partial v} = M_{,xx} - q = 0 \quad (\text{equil. equation})$$



$$\left. \begin{array}{l} \frac{dV}{dx} = q \\ \frac{dM}{dx} = V \end{array} \right\} \frac{d^2 M}{dx^2} = q$$

4.7-7

For a rigid bar, use a linear displacement field.

$$v = [\underline{N}] \{ \underline{d} \} = \left[\frac{L-x}{L} \quad \frac{x}{L} \right] \begin{Bmatrix} v_i \\ v_j \end{Bmatrix}$$

As with a linear spring, strain energy (here the increment dU) is

$$dU = \frac{1}{2} v dF = \frac{1}{2} v (k v dx) = \frac{1}{2} k v^2 dx = \frac{1}{2} k v^T v dx$$

$$U = \frac{1}{2} \int_0^L v^T v k dx = \frac{1}{2} \begin{Bmatrix} v_i \\ v_j \end{Bmatrix}^T \underbrace{\left(\int_0^L [\underline{N}]^T [\underline{N}] k dx \right)}_{[\underline{k}]} \begin{Bmatrix} v_i \\ v_j \end{Bmatrix}$$

$$[\underline{k}] = \frac{k}{L^2} \int_0^L \begin{bmatrix} (L-x)^2 & x(L-x) \\ x(L-x) & x^2 \end{bmatrix} dx = \frac{k}{L^2} \begin{bmatrix} L^3/3 & L^3/6 \\ L^3/6 & L^3/3 \end{bmatrix} = \frac{kL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

4.8-1

$$(a) \Pi_p = \int_0^L \frac{1}{2} EI v_{,xx}^2 dx - \int_0^L q v dx$$

$$v = [N] \{d\}, \quad v_{,xx} = [N_{,xx}] \{d\}$$

$$\Pi_p = \frac{1}{2} \{d\}^T \underbrace{\int_0^L [N_{,xx}]^T EI [N_{,xx}] dx}_{[k]} \{d\} - \underbrace{\{d\}^T \int_0^L [N]^T q dx}_{\{r_q\}}$$

$$\left\{ \frac{\partial \Pi_p}{\partial d} \right\} = \{0\} \text{ yields } [k] \{d\} = \{r_q\}$$

(b) Henceforth $\sim$ denotes a matrix.

$$\Pi = \int_0^L \left(\frac{M^2}{2EI} + M_{,x} v_{,x} + q v \right) dx \quad \begin{matrix} M = \tilde{N}_m \tilde{m}_e \\ v = \tilde{N}_v \tilde{v}_e \end{matrix}$$

$$\begin{aligned} \Pi = & \frac{1}{2} \tilde{m}_e^T \underbrace{\int_0^L \tilde{N}_m^T \tilde{N}_m \frac{dx}{EI}}_{H_{11}} \tilde{m}_e + \underbrace{\tilde{m}_e^T \int_0^L \tilde{N}_{m,x} \tilde{N}_{v,x} dx}_{H_{12}} \tilde{v}_e \\ & + \underbrace{\tilde{v}_e^T \int_0^L \tilde{N}_v^T q dx}_{r_q} \end{aligned}$$

Same as

$$\tilde{v}_e^T H_{12}^T \tilde{m}_e$$

$$\Pi = \frac{1}{2} \tilde{m}_e^T H_{11} \tilde{m}_e + \tilde{m}_e^T H_{12} \tilde{v}_e + \tilde{v}_e^T r_q$$

$$\left\{ \frac{\partial \Pi}{\partial \tilde{m}_e} \right\} = \{0\} = H_{11} \tilde{m}_e + H_{12} \tilde{v}_e = 0$$

$$\left\{ \frac{\partial \Pi}{\partial \tilde{v}_e} \right\} = \{0\} = H_{12}^T \tilde{m}_e + r_q = 0$$

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{12}^T & 0 \end{bmatrix} \begin{Bmatrix} \tilde{m}_e \\ \tilde{v}_e \end{Bmatrix} = \begin{Bmatrix} 0 \\ -r_q \end{Bmatrix}$$

$\tilde{v}_e, p_{,x} = \tilde{N}_{,x} p_e$, etc.

$$\Pi = p_e^T \int_0^L \left(\tilde{N}_{,x}^T \tilde{N}_{,x} + \tilde{N}_{,y}^T \tilde{N}_{,y} + \tilde{N}_{,z}^T \tilde{N}_{,z} \right) dV p_e - \omega^2 p_e^T \int_0^L \frac{1}{c^2} \tilde{N}^T \tilde{N} dV p_e$$

$$\Pi = p_e^T H p_e - \omega^2 p_e^T Q p_e$$

$$\left\{ \frac{\partial \Pi}{\partial p_e} \right\} = \{0\} \text{ yields } (H - \omega^2 Q) p_e = 0$$

$$(d) \Pi_p = \iiint \left(\frac{1}{2} \tilde{\kappa}^T D \tilde{\kappa} - p w \right) dx dy \quad \tilde{\kappa} = \begin{Bmatrix} w_{,xx} \\ w_{,yy} \\ 2w_{,xy} \end{Bmatrix} \quad D = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

$$w = \tilde{N} d \quad \tilde{\kappa} = \begin{Bmatrix} \tilde{N}_{,xx} \\ \tilde{N}_{,yy} \\ 2\tilde{N}_{,xy} \end{Bmatrix} d = \tilde{B} d$$

$$\Pi_p = \frac{1}{2} d^T \underbrace{\iiint \tilde{B}^T D \tilde{B} dx dy}_{[k]} d - d^T \underbrace{\iiint \tilde{N}^T q dx dy}_{[r]}$$

$$\frac{\partial \Pi_p}{\partial d} = 0 \text{ yields } [k] d = [r]$$

4.9-1

$$\theta = 30^\circ \quad \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} a_1 & -0.134 & -0.5 \\ a_4 & 0.5 & -0.134 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

Node 1: $x=y=0$

Node 2: $x=1, y=0$

Node 3: $x=0, y=1$

$$\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} = \begin{Bmatrix} a_1 \\ a_4 \end{Bmatrix}$$

$$\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} a_1 - 0.134 \\ a_4 + 0.5 \end{Bmatrix}$$

$$\begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} a_1 - 0.5 \\ a_4 - 0.134 \end{Bmatrix}$$

Final positions:

$$\begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} = \begin{Bmatrix} a_1 \\ a_4 \end{Bmatrix}, \quad \begin{Bmatrix} x_2 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} a_1 + 0.866 \\ a_4 + 0.5 \end{Bmatrix}, \quad \begin{Bmatrix} x_3 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} a_1 - 0.5 \\ a_4 + 0.866 \end{Bmatrix}$$

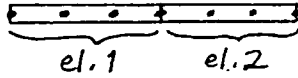
$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{0.866^2 + 0.5^2} = 1 \quad \checkmark$$

$$\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2} = \sqrt{1.366^2 + 0.366^2} = 1.414 = \sqrt{2} \quad \checkmark$$

$$\sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2} = \sqrt{0.5^2 + 0.866^2} = 1 \quad \checkmark$$

4.9-2

Π_p contains 2nd derivatives of v
so $m=2$ and continuity of
first derivatives (v_x) is required between
elements (requirement 2). This the
proposed element cannot do. Also, imagine
two adjacent els.



If all d.o.f. of el. 1
suppressed, but not all d.o.f. of el. 2, then
 $v_x = 0$ in el. 1 but not in el. 2 @ juncture.

4.9-3

Do not want to favor one coord.
direction over another.

Total d.o.f.	Cubic d.o.f.	
11	1	xyz
13	3	x^3, y^3, z^3
14	4	x^3, y^3, z^3, xyz
16	6	$x^2y, xy^2, y^2z, yz^2, z^2x, zx^2$
17	7	$x^2y, xy^2, y^2z, yz^2, z^2x, zx^2,$ xyz
19	9	$x^3, y^3, z^3,$ $x^2y, xy^2, y^2z, yz^2, z^2x, zx^2$ (i.e. all but xyz)

4.9-4

(a) If consistent nodal loads are used, both meshes give exact displacements at nodes (see Ref. 2.5). Between nodes, both meshes give approximate displacements (and stresses). But since linear elements give only straight-line plots of displacement versus axial coordinate, quadratic elements should be able to provide a better fit to displacements produced by smoothly-varying loads (note that if loads do not vary smoothly, it is not obvious which kind of element will be better — for example, a concentrated load applied to the interior node of a quadratic element will produce poor results). Similar remarks can be made for stress fields calculated from element displacement gradients.

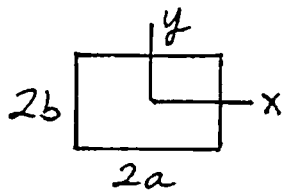
(b) Apply Eqs. 4.9-2, 4.9-3, 4.9-5. Let h = element length, whether linear or quadratic. Orders of error:

	<u>Linear element</u>	<u>Quadratic element</u>
Displacement error	$O(h^2)$	$O(h^3)$
Stress error	$O(h)$	$O(h^2)$

Therefore, when element lengths are halved, approximate reduction factors for error are

	<u>Linear element</u>	<u>Quadratic element</u>
Displacement	$\frac{1}{4}$	$\frac{1}{8}$
Stress	$\frac{1}{2}$	$\frac{1}{4}$

4.10-1



Uniform t , five β 's, $\nu = 0$

$$[\underline{E}] = E \begin{bmatrix} 1 & 1 & 1/2 \end{bmatrix}, \quad [\underline{E}]^{-1} = \frac{1}{E} \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$$

$$[\underline{P}]^T [\underline{E}]^{-1} [\underline{P}] = \frac{1}{E} \begin{bmatrix} 1 & 0 & 0 \\ y & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & y & 0 & 0 & 0 \\ 0 & 0 & 1 & x & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$= \frac{1}{E} \begin{bmatrix} 1 & y & 0 & 0 & 0 \\ y & y^2 & 0 & 0 & 0 \\ 0 & y & 1 & x & 0 \\ 0 & 0 & x & x^2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\int dA = 4ab, \quad \int x dA = \int y dA = 0, \quad \int x^2 dA = \frac{2b(2a)^3}{12} = \frac{4a^3b}{3}$$

$$\int y^2 dA = \frac{2a(2b)^3}{12} = \frac{4ab^3}{3}$$

$$[H] = \int [\underline{P}]^T [\underline{E}]^{-1} [\underline{P}] dV = \frac{4abt}{3} \begin{bmatrix} 1 & \frac{b^3}{3} & 1 & \frac{a^3}{3} & 2 \end{bmatrix}$$

4.10-2

$$M = \begin{bmatrix} 1 & x \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix} \quad \text{or} \quad M = [\tilde{P}] \{ \tilde{\beta} \} \quad \text{where} \quad [\tilde{P}] = \begin{bmatrix} 1 & x \end{bmatrix}$$

$$[\tilde{H}] = \frac{1}{EI_2} \int_0^L \begin{Bmatrix} 1 \\ x \end{Bmatrix} \begin{bmatrix} 1 & x \end{bmatrix} dx = \frac{1}{EI_2} \begin{bmatrix} L & L^2/2 \\ L^2/2 & L^3/3 \end{bmatrix}$$

$$[\tilde{H}]^{-1} = \frac{12EI_2}{L^4} \begin{bmatrix} L^3/3 & -L^2/2 \\ -L^2/2 & L \end{bmatrix} = \frac{12EI_2}{L^3} \begin{bmatrix} L^2/3 & -L/2 \\ -L/2 & 1 \end{bmatrix}$$

$$\text{As in Eq. 4.10-19, } [\tilde{R}] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & L \end{bmatrix}, \quad [\tilde{G}] = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & L \end{bmatrix}$$

$$[\tilde{H}]^{-1} [\tilde{G}] = \frac{12EI_2}{L^3} \begin{bmatrix} -L/2 & -L^2/3 & L/2 & -L^2/6 \\ 1 & L/2 & -1 & L/2 \end{bmatrix}$$

$$[\tilde{k}] = [\tilde{G}]^T [\tilde{H}]^{-1} [\tilde{G}] = \frac{12EI_2}{L^3} \begin{bmatrix} 1 & L/2 & -1 & L/2 \\ L/2 & L^2/3 & -L/2 & L^2/6 \\ -1 & -L/2 & 1 & -L/2 \\ L/2 & L^2/6 & -L/2 & L^2/3 \end{bmatrix}$$

5.1-1

$$\Pi_P = \int_0^{L_T} \left(\frac{1}{2} AE \tilde{u}_{,x}^2 - c x \tilde{u} \right) dx - P u_L$$

Assume $\tilde{u} = a_1 x + a_2 x^2$, then $\tilde{u}_{,x} = a_1 + 2a_2 x$

$$\tilde{u}_{,x}^2 = a_1^2 + 4a_1 a_2 x + 4a_2^2 x^2$$

$$\begin{aligned} \Pi_P = \frac{AE}{2} \left(a_1^2 L_T + 2a_1 a_2 L_T^2 + \frac{4}{3} a_2^2 L_T^3 \right) - c \left(a_1 \frac{L_T^3}{3} + a_2 \frac{L_T^4}{4} \right) \\ - P(a_1 L_T + a_2 L_T^2) \end{aligned}$$

$$\frac{\partial \Pi_P}{\partial a_1} = 0 = AE(a_1 L_T + a_2 L_T^2) - \frac{c L_T^3}{3} - P L_T$$

$$\frac{\partial \Pi_P}{\partial a_2} = 0 = AE \left(a_1 L_T + \frac{4}{3} a_2 L_T^3 \right) - \frac{c L_T^4}{4} - P L_T^2$$

$$\begin{bmatrix} 1 & L_T \\ L_T & \frac{4}{3} L_T^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \frac{c L_T^3}{3AE} + \frac{P}{AE} \\ \frac{c L_T^4}{4AE} + \frac{P L_T}{AE} \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{3}{L_T^2} \begin{bmatrix} \frac{4}{3} L_T^2 & -L_T \\ -L_T & 1 \end{bmatrix} \begin{Bmatrix} \frac{c L_T^3}{3AE} + \frac{P}{AE} \\ \frac{c L_T^4}{4AE} + \frac{P L_T}{AE} \end{Bmatrix} = \begin{Bmatrix} \frac{7c L_T^2}{12AE} + \frac{P}{AE} \\ -\frac{c L_T}{4AE} \end{Bmatrix}$$

$$\tilde{u} = \frac{P}{AE} x + \frac{7c L_T^2}{12AE} x - \frac{c L_T}{4AE} x^2$$

same as Eq. 5.1-10

5.1-2

$$\int_0^L W[EI \tilde{v}_{xxxx} - q] dx = 0, \quad \tilde{v} = ax(L-x)$$

Integrate by parts twice: W

$$\int_0^L W_x[-EI \tilde{v}_{xxx}] dx - \int_0^L Wq dx + EI[W \tilde{v}_{xxx}]_0^L = 0$$

But $W = 0$ at $x=0$ and at $x=L$: $[W \tilde{v}_{xxx}]_0^L = 0$

$$\int_0^L W_{xx}[EI \tilde{v}_{xx}] dx - \int_0^L Wq dx - EI[W_x \tilde{v}_{xx}]_0^L = 0$$

But, as nonessential boundary condition, ends are simply supported; $\tilde{v}_{xx} = 0$ at ends.

Also $\tilde{v}_{xx} = -2a$ and $W_{xx} = -2$, so

$$\int_0^L -2EI(-2a) dx - \int_0^L x(L-x)q dx = 0 \text{ and}$$

$$a = \frac{1}{4EIL} \int_0^L x(L-x)q dx; \text{ center } \tilde{v} = a \frac{L^2}{4}$$

(a) $q = q_0 \sin \frac{\pi x}{L}$. Let $\theta = \frac{\pi x}{L}$; then

$$a = \frac{1}{4EIL} \int_0^L x(L-x) q_0 \sin \frac{\pi x}{L} dx \text{ becomes}$$

$$a = \frac{L^2 q_0}{EI \pi^2} \int_0^\pi \theta \left(1 - \frac{\theta}{\pi}\right) \sin \theta d\theta = 0.03225 \frac{q_0 L^2}{EI}$$

$$\text{At center, } \tilde{v} = a \frac{L^2}{4} = 0.00806 \frac{q_0 L^4}{EI}$$

Exact $v = v_c \sin \frac{\pi x}{L}$ where $v_c = \text{center } v$.

$$EI v_{xxxx} = q, \quad EI \frac{\pi^4}{L^4} v_c \sin \frac{\pi x}{L} = q_0 \sin \frac{\pi x}{L}$$

$$a. L^4 \quad 0.01027 \frac{q_0 L^4}{EI}$$

Approx. center deflection is 21.5% low.

$M = EI v_{xx}$: at center,

$$\text{exact } M_c = EI \frac{q_0 L^4}{\pi^4 EI} \left(-\frac{\pi^2}{L^2}\right)$$

$$= 0.1013 q_0 L^2$$

$$\text{Approx. } M = EI(-2a) = 0.0645 q_0 L^2$$

(36% low)

$$a = \frac{q_0}{4EIL} \int_0^L x(L-x) dx = \frac{q_0}{4EIL} \frac{L^2}{4} \frac{2L}{3} = \frac{q_0 L^2}{24EI}$$

$$\text{At center, } \tilde{v} = a \frac{L^2}{4} = \frac{q_0 L^4}{96EI} = 0.01042 \frac{q_0 L^4}{EI}$$

$$\text{Exact is } \frac{5 q_0 L^4}{384EI} = 0.01302 \frac{q_0 L^4}{EI}$$

Approx. center deflection is 20.0% low.

$$\text{At center, exact } M_c = \frac{q_0 L^2}{8}$$

$$\text{Approx. } M = EI(-2a) = \frac{q_0 L^2}{12}$$

(33% low)

5.2-1

At $x = \frac{L_T}{3}$, from collocation:

$$\tilde{u} = \left(\frac{P}{AE} + \frac{cL_T^2}{2AE} \right) \frac{L_T}{3} - \frac{cL_T}{6AE} \frac{L_T^2}{9} = \frac{PL_T}{3AE} - \frac{4cL_T^3}{27AE}$$

At $x = \frac{L_T}{3}$, exact solution (Eq. 5.1-4):

$$u = \frac{PL_T}{3AE} + \frac{cL_T^2}{2AE} \frac{L_T}{3} - \frac{c}{6AE} \frac{L_T^3}{27} = \frac{PL_T}{3AE} - \frac{4.33cL_T^3}{27AE}$$

That $\tilde{u} \neq u$ at the collocation point does not imply an error. The residual depends on derivatives of $\tilde{u}$. Making the residual vanish at a certain point does not imply that the dependent variable is exact at that point.

5.2-2

$$\frac{d^2 u}{dx^2} + \frac{c}{AE} = 0 \quad \text{in } V$$

$$\frac{du}{dx} = \frac{P}{AE} \quad \text{at } x = L_T$$

Assume $\tilde{u} = a_1 x + a_2 x^2$

then $R = 2a_2 + \frac{c}{AE}$

$$R_B = a_1 + 2a_2 L_T - \frac{P}{AE}$$

Collocation Point doesn't matter; $a_2 = -\frac{c}{2AE}$ from $R=0$

$$a_1 - \frac{cL_T}{AE} - \frac{P}{AE} = 0; \quad a_1 = \frac{P}{AE} + \frac{cL_T}{AE} \quad \text{from } R_B = 0$$

$$u = \frac{Px}{AE} + \frac{c}{AE} \left(L_T x - \frac{x^2}{2} \right) \quad [\text{exact}]$$

Subdomain $\int_0^{L_T} \left(2a_2 + \frac{c}{AE} \right) dx = 0; \quad a_2 = -\frac{c}{2AE}$ Same results as above.

Least Squares ($\alpha = \frac{1}{L_T}$)

$$I = \int_0^{L_T} \left(2a_2 + \frac{c}{AE} \right)^2 dx + \frac{1}{L_T} \left[a_1 + 2a_2 L_T - \frac{P}{AE} \right]^2$$

$$I = \left(2a_2 + \frac{c}{AE} \right)^2 L_T + \frac{1}{L_T} \left[a_1 + 2a_2 L_T - \frac{P}{AE} \right]^2$$

$$\frac{\partial I}{\partial a_1} = \frac{2}{L_T} \left(a_1 + 2a_2 L_T - \frac{P}{AE} \right) = 0, \quad \frac{\partial I}{\partial a_2} = 2L_T \left(2a_2 + \frac{c}{AE} \right) + \frac{2}{L_T} \left[\dots \right] 2L_T = 0$$

$$\frac{2}{L_T} a_1 + 4a_2 = \frac{2P}{AEL_T}$$

$$4a_1 + 16L_T a_2 = \frac{4P}{AE} - \frac{4cL_T}{AE}$$

Same results as above.

Least Squares Collocation (at $x = \frac{L_T}{3}$ and $x = L_T$ for R , at $x = L$ for R_B . Also set $\alpha = 1/L_T$.)

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 2 \\ 1/L_T & 2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} - \begin{Bmatrix} -c/AE \\ -c/AE \\ P/AEL_T \end{Bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1/L_T \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 2 \\ 1/L_T & 2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1/L_T \\ 2 & 2 & 2 \end{bmatrix} \begin{Bmatrix} -c/AE \\ -c/AE \\ P/AEL_T \end{Bmatrix}$$

$$\begin{bmatrix} 1/L_T^2 & 2/L_T \\ 2/L_T & 12 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} P/AEL_T^2 \\ -4c/AE + 2P/AEL_T \end{Bmatrix} \quad \text{Same results as above}$$

[continued, next page]

5.2-2 (continued) Galerkin

$$R_i = \int_0^{L_T} \left(-\frac{dW_i}{dx} \frac{d\tilde{u}}{dx} + W_i \frac{c}{AE} \right) dx + W_i \frac{P}{AE} \Big|_{L_T} \quad \text{where } \begin{matrix} W_1 = x \\ W_2 = x^2 \end{matrix}$$

$$0 = \int_0^{L_T} \left[(-1)(a_1 + 2a_2x) + x \frac{c}{AE} \right] dx + \frac{PL_T}{AE}$$

$$0 = \int_0^{L_T} \left[(-2x)(a_1 + 2a_2x) + x^2 \frac{c}{AE} \right] dx + \frac{PL_T^2}{AE}$$

$$0 = -L_T a_1 - L_T^2 a_2 + \frac{cL_T^2}{2AE} + \frac{PL_T}{AE}$$

$$0 = -L_T^2 a_1 - \frac{4}{3} L_T^3 a_2 + \frac{cL_T^3}{3} + \frac{PL_T^2}{AE}$$

$$\begin{bmatrix} 1 & L_T \\ L_T & 4L_T^2/3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} cL_T/2AE + P/AE \\ cL_T^2/3AE + PL_T/AE \end{Bmatrix} \quad \text{Same results as before}$$

5.2-3

Exact values: At $x=0.5$, $u = 1.1492$

At $x=0.7$, $u = 0.8325$

Residual methods:

$$\tilde{u} = 3 - 2x + a(x^2 - x), \quad \tilde{u}_{,x} = -2 + a(2x - 1),$$

$$\tilde{u}_{,xx} = 2a, \quad R = 4ax^2 - (4a + 8)x + 2a$$

(a) $R=0$ at $x = \frac{1}{2}$ yields $a = 4$
and $\tilde{u} = 4x^2 - 6x + 3$

At $x=0.5$, $\tilde{u} = 1.00$; -13.0% error

At $x=0.7$, $\tilde{u} = 0.76$; -8.7% error

(b) $\int_0^1 R dx = \frac{4a}{3} - (2a + 4) + 2a = 0$ yields
 $a = 3$ and $\tilde{u} = 3x^2 - 5x + 3$

At $x=0.5$, $\tilde{u} = 1.25$; $+8.8\%$ error

At $x=0.7$, $\tilde{u} = 0.97$; $+16.5\%$ error

(c) $\frac{\partial}{\partial a} \int_0^1 R^2 dx = 0, \quad \int_0^1 R \frac{\partial R}{\partial a} dx = 0$

$\int_0^1 [4ax^2 - (4a + 8)x + 2a](4x^2 - 4x + 2) dx = 0$
yields $a = \frac{20}{7}$, $\tilde{u} = \frac{20}{7}x^2 - \frac{34}{7}x + 3$

At $x=0.5$, $\tilde{u} = 1.2857$; $+11.9\%$ error

At $x=0.7$, $\tilde{u} = 1.0000$; $+20.1\%$ error

(d) $I = \left[\frac{4a}{9} - \frac{4a}{3} - \frac{8}{3} + 2a \right]^2 + \left[\frac{16a}{9} - \frac{8a}{3} - \frac{16}{3} + 2a \right]^2$
 $I = \frac{200a^2}{81} - \frac{160a}{9} + \frac{320}{9}, \quad \frac{\partial I}{\partial a} = \frac{400a}{81} - \frac{160}{9}$
 $\frac{\partial I}{\partial a} = 0$ yields $a = \frac{18}{5}$, $\tilde{u} = \frac{18}{5}x^2 - \frac{28}{5}x + 3$

At $x=0.5$, $\tilde{u} = 1.100$; -4.3% error

At $x=0.7$, $\tilde{u} = 0.844$; $+1.4\%$ error

(e) $\frac{\partial \tilde{u}}{\partial a} = x^2 - x, \quad \int_0^1 (x^2 - x) R dx = 0$

yields $a = \frac{10}{3}$, $\tilde{u} = \frac{10}{3}x^2 - \frac{16}{3}x + 3$

At $x=0.5$, $\tilde{u} = 1.1667$; $+1.5\%$ error

At $x=0.7$, $\tilde{u} = 0.9000$; $+8.1\%$ error

5.2-4

Exact values: At $x=0.5$, $u=1.4715$

Residual methods: At $x=0.7$, $u=2.5864$

$$\tilde{u} = a_1 x + a_2 x^2, \quad \tilde{u}_{,x} = a_1 + 2a_2 x$$

$$R = \tilde{u}_{,x} + 2\tilde{u} - 16x = (1+2x)a_1 + 2x(1+x)a_2 - 16x$$

$$(a) \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{bmatrix} 1.5 & 0.625 \\ 2.0 & 1.500 \\ 2.5 & 2.625 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} - \begin{Bmatrix} 4 \\ 8 \\ 12 \end{Bmatrix} = [\underline{Q}]\{\underline{a}\} - \{\underline{c}\}$$

$$\text{Apply Eq. 5.2-13c: } \begin{bmatrix} 12.5 & 10.5 \\ 10.5 & 9.5313 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 52 \\ 46 \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 1.4202 \\ 3.2616 \end{Bmatrix}, \quad \tilde{u} = 1.4202x + 3.2616x^2$$

At $x=0.5$, $\tilde{u} = 1.5255$; +3.7% error

At $x=0.7$, $\tilde{u} = 2.5923$; +0.2% error

$$(b) \frac{\partial \tilde{u}}{\partial a_1} = x, \quad \frac{\partial \tilde{u}}{\partial a_2} = x^2; \text{ set } \int_0^1 x R dx = \int_0^1 x^2 R dx = 0$$

Thus

$$\left. \begin{aligned} \left(\frac{1}{2} + \frac{2}{3}\right)a_1 + \left(\frac{2}{3} + \frac{1}{2}\right)a_2 - \frac{16}{3} &= 0 \\ \left(\frac{1}{3} + \frac{1}{2}\right)a_1 + \left(\frac{1}{2} + \frac{2}{5}\right)a_2 - 4 &= 0 \end{aligned} \right\} \begin{aligned} a_1 &= 1.7143 \\ a_2 &= 2.8571 \end{aligned}$$

At $x=0.5$, $\tilde{u} = 1.5714$; +6.8% error

At $x=0.7$, $\tilde{u} = 2.5429$; -1.7% error

5.2-5 $\tilde{u} = a_1 x + a_2 x^2$ $R = 2a_2 + cx$
 $R_S = (a_1 + 2a_2 L_T) - b$

(a) $R = 0 = 2a_2 + c \frac{L_T}{2}$,
 $R_S = 0$ yields $a_1 = b + \frac{c L_T}{2}$
 For $b=0$ & $c=L_T=1$, $\tilde{u} = \frac{x}{2} - \frac{x^2}{4}$, $\tilde{u}_{,x} = \frac{1}{2} - \frac{x}{2}$
 (b) $\int_0^{L_T/2} (2a_2 + cx) dx = 2a_2 \frac{L_T}{2} + \frac{c}{2} \left(\frac{L_T}{2}\right)^2 = 0$, $a_2 = -\frac{c L_T}{8}$
 $R_S = 0$ yields $a_1 = b + \frac{c L_T}{4}$
 For $b=0$ & $c=L_T=1$, $\tilde{u} = \frac{x}{4} - \frac{x^2}{8}$, $\tilde{u}_{,x} = \frac{1}{4} - \frac{x}{4}$

(c) $\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_S \end{Bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 2 \\ 0 & 2 \\ 1/L_T & 2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} - \begin{Bmatrix} 0 \\ -c L_T/2 \\ -c L_T \\ b/L_T \end{Bmatrix}$
 $\begin{bmatrix} \frac{1}{L_T^2} & \frac{2}{L_T} \\ \frac{2}{L_T} & 1/b \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \frac{b}{L_T^2} \\ -\frac{3c}{L_T} + \frac{2b}{L_T} \end{Bmatrix}$, $\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} b + \frac{c L_T}{2} \\ -\frac{c L_T}{4} \end{Bmatrix}$

For $b=0$ & $c=L_T=1$, $\tilde{u} = \frac{x}{2} - \frac{x^2}{4}$, $\tilde{u}_{,x} = \frac{1}{2} - \frac{x}{2}$

(d) Becomes simple collocation

$\begin{bmatrix} 0 & 2 \\ 1/L_T & 2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} - \begin{Bmatrix} -c L_T/3 \\ b/L_T \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$, $a_1 = b + \frac{c L_T}{3}$, $a_2 = -\frac{c L_T}{6}$

For $b=0$ & $c=L_T=1$, $\tilde{u} = \frac{x}{3} - \frac{x^2}{6}$, $\tilde{u}_{,x} = \frac{1}{3} - \frac{x}{3}$

Summary of foregoing results:

	(a)	(b)	(c)	(d)	(exact)
$u @ L_T/2$	.1875	.0938	.1875	.1250	.2292
$u @ L_T$	.2500	.1250	.2500	.1667	.3333
$u @ \wedge$	.5000	.2500	.5000	.3333	.5000
$u_{,x} @ L_T/2$	.2500	.1250	.2500	.1667	.3750
$u_{,x} @ L_T$	0	0	0	0	0

(e) Consider n residuals R_i . Let $c_i = \text{constants}$. Form of Eq. 15.3-7, becomes

$\{R\} = \begin{Bmatrix} R_1 \\ R_2 \\ \vdots \\ R_S \end{Bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 2 \\ \vdots & \vdots \\ \alpha & 2\alpha \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} - \begin{Bmatrix} c_1 \\ c_2 \\ \vdots \\ \alpha b \end{Bmatrix}$

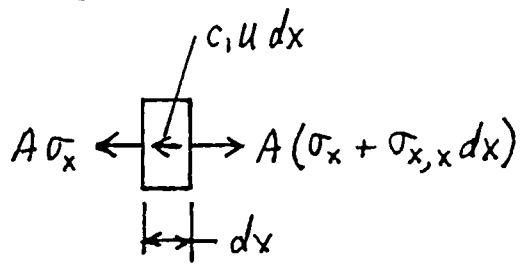
Form of Eq. 15.3-8c, & its solution, are

$\begin{bmatrix} \alpha^2 & 2\alpha^2 \\ 2\alpha^2 & 4n+4\alpha^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \alpha^2 b \\ 2\sum c_i + 2\alpha^2 b \end{Bmatrix}$

$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{1}{4n\alpha^2} \begin{bmatrix} 4n+4\alpha^2 & -2\alpha^2 \\ -2\alpha^2 & \alpha^2 \end{bmatrix} \begin{Bmatrix} \alpha^2 b \\ 2\sum c_i + 2\alpha^2 b \end{Bmatrix} = \begin{Bmatrix} b - \frac{1}{n} \sum c_i \\ \frac{1}{2n} \sum c_i \end{Bmatrix}$

This result is independent of α due to the fortuitous absence of α from interior residuals.

5.3-1



Axial equilibrium:

$$\left. \begin{aligned} A\sigma_{x,x} - c_1 u &= 0 \\ \sigma_x &= E u_{,x} \end{aligned} \right\} AE u_{,xx} - c_1 u = 0$$

Assume $\tilde{u} = [N] \{d\} = \left[\frac{L-x}{L} \quad \frac{x}{L} \right] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$ over one element of length L

$$R=0 = \int_0^L [N]^T (AE \tilde{u}_{,xx} - c_1 \tilde{u}) dx$$

Substitute for $\tilde{u}$ and integrate by parts

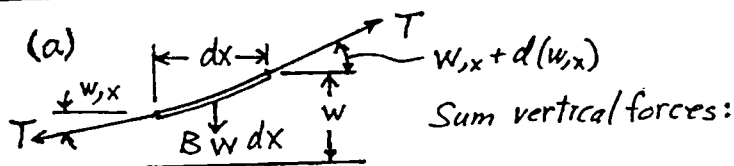
$$0 = \int_0^L [N_{,x}]^T [N_{,x}] AE dx \{d\} + \int_0^L [N]^T [N] c_1 dx \{d\} - \left[[N]^T AE \tilde{u}_{,x} \right]_0^L$$

$$\left(\frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + c_1 L \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \right) \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} (-AE \tilde{u}_{,x})_0 \\ (AE \tilde{u}_{,x})_L \end{Bmatrix}$$

$$\left(\frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_1 L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right) \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P \\ -F \end{Bmatrix}$$

where $F = c_2 u_L$

5.3-2



$$-Tw_{,x} + T[w_{,x} + d(w_{,x})] - Bwdx = 0$$

$$Td(w_{,x}) = Bwdx, \quad Tw_{,xx} - Bw = 0$$

(b) $\tilde{w} = [N]\{d\}$ and $\sum \int_0^L [N]^T (T\tilde{w}_{,xx} - B\tilde{w}) dx = 0$

Integrate by parts

$$\sum \int_0^L (-[N_{,x}]^T T \tilde{w}_{,x} - [N]^T B \tilde{w}) dx + \sum [N]^T T \tilde{w}_{,x} \Big|_0^L$$

$\tilde{w}_{,x}$ at $x=0$ $\tilde{w}_{,x}$ at right end F_L & F_R enter the last term

Also substitute $\tilde{w} = [N]\{d\}$ & $\tilde{w}_{,x} = [N_{,x}]\{d\}$

$$\sum \left(\underbrace{\int_0^L [N_{,x}]^T T [N_{,x}] dx}_{[k]} + \underbrace{\int_0^L [N]^T B [N] dx}_{[k_f]} \right) \{d\} = \begin{Bmatrix} F_L \\ \vdots \\ F_R \end{Bmatrix}$$

5.3-3

$$(a) \int_0^L [N] \left(T \tilde{w}_{,xx} - \rho_L \ddot{w} \right) dx = 0$$

Also, $\tilde{w} = [N] \{d\}$. Integrate by parts.

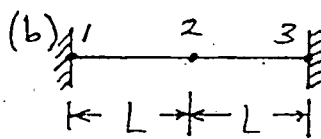
$$-\int_0^L [N_{,x}]^T T \tilde{w}_{,x} dx - \int_0^L [N]^T \rho_L \ddot{w} dx + \left[[N]^T T \tilde{w}_{,x} \right]_0^L = 0$$

Last term associated with transverse support forces, whose d.o.f. are discarded for simply supported ends. Subs: $\tilde{w} = [N] \{d\}$, $\tilde{w}_{,x} = [N_{,x}] \{d\}$

$$\underbrace{\int_0^L [N_{,x}]^T [N_{,x}] dx}_{[k]} \{d\} + \underbrace{\int_0^L [N]^T \rho_L [N] dx}_{[m]} \{\ddot{d}\} = 0$$

Here $[N] = \begin{bmatrix} \frac{L-x}{L} & \frac{x}{L} \end{bmatrix}$ and $[N_{,x}] = \frac{1}{L} \begin{bmatrix} -1 & 1 \end{bmatrix}$

$$[k] = \frac{T}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad [m] = \frac{\rho_L L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

(b)  Assemble two elements & suppress d.o.f. w_1 & w_3 .
Thus $2 \frac{T}{L} w_2 + 4 \frac{\rho_L L}{6} \ddot{w}_2 = 0$

Set $w_2 = \bar{w}_2 \sin \omega t$

$$\ddot{w}_2 = -\omega^2 \bar{w}_2 \sin \omega t \quad \left(2 \frac{T}{L} + 4 \frac{\rho_L L}{6} \omega^2 \right) \bar{w}_2 = 0$$

$$\omega^2 = 3 \frac{T}{\rho_L L^2}$$

5.3-4

Left hand side of Eq. 5.3-18, after integration by parts of the term that contains $F \tilde{w}_{,xx}$, contains the additional terms

$$+ \int_0^L [N_{,x}]^T F \tilde{w}_{,x} dx + \int_0^L [N]^T B \tilde{w} dx$$

Substitute $\tilde{w} = [N] \{d\}$ and $\tilde{w}_{,x} = [N_{,x}] \{d\}$

The additional terms become

$$+ \underbrace{\int_0^L [N_{,x}]^T F [N_{,x}] dx}_{[k_\sigma]} \{d\} + \underbrace{\int_0^L [N]^T B [N] dx}_{[k_f]} \{d\}$$

5.3-5

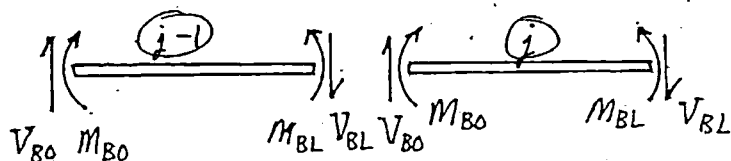
We want to show that the load terms are $\{R\} = ([N, x]^T M_B - [N]^T V_B)_0^L$

(with assembly of elements implied).

Use cubic shape functions and insert limits

0 to L: $\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} M_{BL} - \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} M_{B0} - \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix} V_{BL} + \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} V_{B0}$

Consider adjacent elements $j-1$ and j .



Consider e.g. assembly of vertical forces where elements $j-1$ and j meet. The dashed line connects forces having the same global d.o.f. number. These add,

to yield the net force $(V_{B0})_j - (V_{BL})_{j-1}$ at the shared node.

$\begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix} - V_{BL} \quad \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + V_{B0}$

5.3-6

$A = \frac{L-x}{L} A_1 + \frac{x}{L} A_2$. Use Eq. 5.3-23.

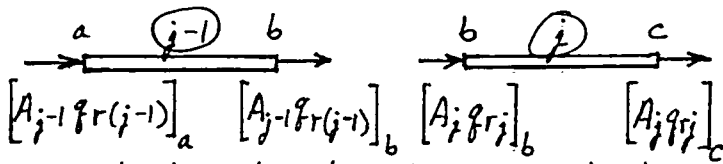
$[N, x]^T [N, x] = \frac{1}{L^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, so integral is

$\frac{k}{L^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_0^L \left(\frac{L-x}{L} A_1 + \frac{x}{L} A_2 \right) dx =$
 $\frac{k}{L^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \left(\frac{A_1 L}{2} + \frac{A_2 L}{2} \right) = \frac{k(A_1 + A_2)}{2L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Eq. 5.3-24 becomes: $\frac{k(A_1 + A_2)}{2L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} A_1 q_1 \\ A_2 q_2 \end{Bmatrix}$

5.3-7

Consider adjacent els. $j-1$ and j .



Connect elements at node b . Right-hand side of Eq. 5.3-23 becomes

$$\begin{array}{l} \text{node } a \text{ --- } \left\{ A_{j-1} q_{r(j-1)} \right\} \\ \text{node } b \text{ --- } \left\{ -A_{j-1} q_{r(j-1)} \right\} + \left\{ A_j q_{rj} \right\} = \left\{ \begin{array}{c} A_{j-1} q_{r(j-1)} \\ \Delta \\ -A_j q_{rj} \end{array} \right\} \\ \text{node } c \text{ --- } \left\{ -A_j q_{rj} \right\} \end{array}$$

$$\text{where } \Delta = [A_j q_{rj}]_b - [A_{j-1} q_{r(j-1)}]_b = 0$$

because of interelement continuity. The other two entries are complete if there are no additional elements.

5.5-1

Write $\phi_{,x}^2 = \phi_{,x}^T \phi_{,x}$ and $\phi_{,y}^2 = \phi_{,y}^T \phi_{,y}$

Substitute $\phi = \underline{N} \underline{\phi}_e$, $\phi_{,x} = \underline{N}_{,x} \underline{\phi}_e$, $\phi_{,y} = \underline{N}_{,y} \underline{\phi}_e$

$$\Pi = \frac{1}{2} \underline{\phi}_e^T \iint (\underline{N}_{,x}^T k_x \underline{N}_{,x} + \underline{N}_{,y}^T k_y \underline{N}_{,y}) dx dy \underline{\phi}_e$$

$$- \underline{\phi}_e^T \iint \underline{N}^T Q dx dy - \underline{\phi}_e^T \int \underline{N}^T f_B dS$$

$$\left\{ \frac{\partial \Pi}{\partial \underline{\phi}_e} \right\} = 0 = \iint (\underline{N}_{,x}^T k_x \underline{N}_{,x} + \underline{N}_{,y}^T k_y \underline{N}_{,y}) dx dy \underline{\phi}_e$$

$$- \iint \underline{N}^T Q dx dy - \int \underline{N}^T f_B dS$$

5.5-2

Residual equation is

$$\int_{\tilde{N}} \underbrace{\left(\tilde{P}_{,xx} + \tilde{P}_{,yy} + \tilde{P}_{,zz} + \frac{\omega^2}{c^2} \tilde{P} \right)}_{\nabla^2 \tilde{P}} dV = 0 \quad (A)$$

Integrate by parts (see Eq. 5.5-5, with $k_x = k_y = k_z = 1$). $\tilde{P}_{,n} = 0$

$$\int_{\tilde{N}} \nabla^2 \tilde{P} dV = \int_{\tilde{N}} \left(\tilde{P}_{,x} l + \tilde{P}_{,y} m + \tilde{P}_{,z} n \right) dS - \int_{\tilde{N}} \left(\tilde{N}_{,x}^T \tilde{P}_{,x} + \tilde{N}_{,y}^T \tilde{P}_{,y} + \tilde{N}_{,z}^T \tilde{P}_{,z} \right) dV$$

Hence, with $\tilde{P}_{,x} = \tilde{N}_{,x}^T \underline{P}_e$ etc., Eq. (A) becomes

$$\int_{\tilde{N}} \left(\tilde{N}_{,x}^T \tilde{N}_{,x} + \tilde{N}_{,y}^T \tilde{N}_{,y} + \tilde{N}_{,z}^T \tilde{N}_{,z} \right) dV \underline{P}_e - \omega^2 \int_{\tilde{N}} \frac{1}{c^2} \tilde{N}^T \tilde{N} dV \underline{P}_e = 0$$

5.5-3

Apply integ. by parts. First, use
Eq. 5.4-7 on first term of -

$$\int_{\tilde{V}} N^T \left(\frac{1}{r} (r \tilde{\phi}_{,r})_{,r} + \frac{1}{r^2} \tilde{\phi}_{,\theta\theta} + \tilde{\phi}_{,zz} + \frac{Q}{k} \right) dV = 0$$

Thus

$$\int_{\tilde{V}} N^T \left(\frac{1}{r} (r \tilde{\phi}_{,r})_{,r} \right) dV = - \int_{\tilde{V}} N_{,r}^T \tilde{\phi}_{,r} dV + \int_{\tilde{S}} N^T \tilde{\phi}_{,r} l dS$$

Integration of second term by parts produces no surface integral because $m=0$ for a normal to the boundary.

$$\int_{\tilde{V}} N^T \frac{1}{r^2} \tilde{\phi}_{,\theta\theta} dV = - \int_{\tilde{V}} N_{,\theta}^T \frac{1}{r^2} \tilde{\phi}_{,\theta} dV. \text{ Finally}$$

$$\int_{\tilde{V}} N^T \tilde{\phi}_{,zz} dV = - \int_{\tilde{V}} N_{,z}^T \tilde{\phi}_{,z} dV + \int_{\tilde{S}} N^T \tilde{\phi}_{,z} n dS$$

With the given boundary condition, we have

$$\int_{\tilde{V}} \left(N_{,r}^T \tilde{\phi}_{,r} + N_{,\theta}^T \frac{1}{r^2} \tilde{\phi}_{,\theta} + N_{,z}^T \tilde{\phi}_{,z} \right) k dV + \int_{\tilde{S}} N^T Q dV + \int_{\tilde{S}} N^T f_B dS$$

Substitute $\tilde{\phi}_{,r} = N_{,r} \phi$, etc. Thus

$$\int_{\tilde{V}} \left(N_{,r}^T N_{,r} + \frac{1}{r^2} N_{,\theta}^T N_{,\theta} + N_{,z}^T N_{,z} \right) k dV \phi = \chi$$

$$\text{where } \chi = \int_{\tilde{V}} N^T Q dV + \int_{\tilde{S}} N^T f_B dS$$

Note: $dV = r dr dz$ for a 1-radian segment.

5.5-4

Multiply by weighting function N^T .

$$\iint N^T (\tilde{\psi}_{,xx} + \tilde{\psi}_{,yy} + A\tilde{\psi}_{,x} + B\tilde{\psi}_{,y} + C) dx dy = 0 \quad (A)$$

Integrate first two terms by parts.

$$\iint N^T \tilde{\psi}_{,xx} dx dy = - \iint N_{,x}^T \tilde{\psi}_{,x} dx dy + \int N^T \tilde{\psi}_{,x} l dS$$

$$\iint N^T \tilde{\psi}_{,yy} dx dy = - \iint N_{,y}^T \tilde{\psi}_{,y} dx dy + \int N^T \tilde{\psi}_{,y} m dS$$

Boundary terms: $\tilde{\psi}_{,x} l + \tilde{\psi}_{,y} m = \tilde{\psi}_{,n} = 0$.

Subs. $\tilde{\psi}_{,x} = N_{,x}^T \psi_e$ etc. into what remains of Eq. (A).

$$\iint \left(-N_{,x}^T N_{,x} - N_{,y}^T N_{,y} + A N^T N_{,x} + B N^T N_{,y} \right) dx dy \psi_e = - \iint N^T C dx dy$$

Here the coefficient matrix of nodal d.o.f. ψ_e is not symmetric.

5.5-5

Companions to Eq. 5.5-11 are

$$\iint \tilde{N}^T \tilde{\tau}_{xy,y} dx dy = - \iint \tilde{N}_{,y}^T \tilde{\tau}_{xy} dx dy + \iint \tilde{N}_{xy}^T m dS$$

$$\iint \tilde{N}^T \tilde{\tau}_{xy,x} dx dy = - \iint \tilde{N}_{,x}^T \tilde{\tau}_{xy} dx dy + \iint \tilde{N}_{xy}^T l dS$$

$$\iint \tilde{N}^T \tilde{\sigma}_{y,y} dx dy = - \iint \tilde{N}_{,y}^T \tilde{\sigma}_y dx dy + \iint \tilde{N}_y^T m dS$$

In view of surface-traction Eqs. 5.5-8, Eqs. 5.5-10 now read

$$- \iint \begin{bmatrix} \tilde{N}_{,x}^T & \underline{0}^T & \tilde{N}_{,y}^T \\ \underline{0}^T & \tilde{N}_{,y}^T & \tilde{N}_{,x}^T \end{bmatrix} \begin{Bmatrix} \tilde{\sigma}_x \\ \tilde{\sigma}_y \\ \tilde{\tau}_{xy} \end{Bmatrix} dx dy + \iint \begin{bmatrix} \tilde{N}^T & \underline{0}^T \\ \underline{0}^T & \tilde{N}^T \end{bmatrix} \begin{Bmatrix} F_x \\ F_y \end{Bmatrix} dx dy + \iint \begin{bmatrix} \tilde{N}^T & \underline{0}^T \\ \underline{0}^T & \tilde{N}^T \end{bmatrix} \begin{Bmatrix} \Phi_x \\ \Phi_y \end{Bmatrix} dS = 0$$

Or, using conventional notation,

$$- \iint [\tilde{B}]^T \{\tilde{\sigma}\} dx dy + \iint [\tilde{N}]^T \{F\} dx dy + \iint [\tilde{N}]^T \{\Phi\} dS \quad (A)$$

$$\text{But } \{\tilde{\sigma}\} = [\tilde{E}] (\{\tilde{\epsilon}\} - \{\epsilon_0\}) + \{\sigma_0\}$$

$$\text{and } \{\tilde{\epsilon}\} = [\tilde{B}] \{d\}$$

$$\text{so } \{\tilde{\sigma}\} = [\tilde{E}] [\tilde{B}] \{d\} - [\tilde{E}] \{\epsilon_0\} + \{\sigma_0\} \quad (B)$$

Eqs. (A) and (B) yield the standard eqs.

5.5-6

Consider unit thickness, as usual.

There are only radial displacements, so we need only $\tilde{u} = \tilde{N} \underline{d}$. $dV = 2\pi r dr$.

$$\int \tilde{N}^T \left(\frac{1}{r} \frac{d}{dr} (r \tilde{\sigma}_r) - \frac{\tilde{\sigma}_\theta}{r} + \rho \omega^2 r \right) dV = 0 \quad (A)$$

Apply Eq. 5.4-7 with $l = 1$.

$$\int \tilde{N}^T \left[\frac{1}{r} \frac{d}{dr} (r \tilde{\sigma}_r) \right] dV = - \int \tilde{N}_{,r}^T \tilde{\sigma}_r dV + \int \tilde{N}^T \tilde{\sigma}_r dS$$

Last term vanishes, as $\sigma_r = 0$ @ $r = r_i$ & $r = r_o$.

(A) becomes $\underbrace{\hspace{10em}}_{\{X\}}$

$$\int_{r_i}^{r_o} \left[\tilde{N}_{,r}^T \quad \frac{1}{r} \tilde{N}^T \right] \begin{Bmatrix} \tilde{\sigma}_r \\ \tilde{\sigma}_\theta \end{Bmatrix} 2\pi r dr = \int_{r_i}^{r_o} \tilde{N}^T \rho \omega^2 r (2\pi r) dr$$

$$\begin{Bmatrix} \tilde{\sigma}_r \\ \tilde{\sigma}_\theta \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{Bmatrix} \tilde{\epsilon}_r \\ \tilde{\epsilon}_\theta \end{Bmatrix} = [\underline{E}] \begin{Bmatrix} \tilde{u}_{,r} \\ \frac{\tilde{u}}{r} \end{Bmatrix} = [\underline{E}] \begin{bmatrix} \tilde{N}_{,r} \\ \frac{1}{r} \tilde{N} \end{bmatrix} \{ \underline{d} \}$$

$$\int_{r_i}^{r_o} \left[\tilde{N}_{,r}^T \quad \frac{1}{r} \tilde{N}^T \right] [\underline{E}] \begin{bmatrix} \tilde{N}_{,r} \\ \frac{1}{r} \tilde{N} \end{bmatrix} 2\pi r dr \{ \underline{d} \} = \{ \underline{f} \}$$

5.5-7

Multiply equilibrium eqs.

by weight function N^T and integrate.

$$\int N^T \left(\frac{1}{r} (r \tilde{\sigma}_r)_{,r} + \tilde{\tau}_{rz,z} - \frac{\tilde{\sigma}_\theta}{r} \right) dV = 0 \quad (A)$$

$$\int N^T \left(\frac{1}{r} (r \tilde{\tau}_{rz})_{,r} + \tilde{\sigma}_{z,z} \right) dV = 0 \quad (B)$$

Integrations by parts (e.g. Eq. 5.4-7):

$$\int N^T \left(\frac{1}{r} (r \tilde{\sigma}_r)_{,r} \right) dV = - \int N_{,r}^T \tilde{\sigma}_r dV + \int N^T \tilde{\sigma}_r / r dS$$

$$\int N^T \tilde{\tau}_{rz,z} dV = - \int N_{,z}^T \tilde{\tau}_{rz} dV + \int N^T \tilde{\tau}_{rz} n dS$$

$$\int N^T \left(\frac{1}{r} (r \tilde{\tau}_{rz})_{,r} \right) dV = - \int N_{,r}^T \tilde{\tau}_{rz} dV + \int N^T \tilde{\tau}_{rz} / r dS$$

$$\int N^T \tilde{\sigma}_{z,z} dV = - \int N_{,z}^T \tilde{\sigma}_z dV + \int N^T \tilde{\sigma}_z n dS$$

Now $l \tilde{\sigma}_r + n \tilde{\tau}_{rz} = \tilde{\Phi}_r$ & $l \tilde{\tau}_{rz} + n \tilde{\sigma}_z = \tilde{\Phi}_z$, so the two residual eqs. (A) & (B) become

$$\int \left[\begin{array}{cc} N_{,r}^T & 0^T \\ 0^T & N_{,z}^T \end{array} \right] \left\{ \begin{array}{c} \tilde{\sigma}_r \\ \tilde{\sigma}_z \\ \tilde{\tau}_{rz} \end{array} \right\} dV = \int \left[\begin{array}{cc} N^T & 0^T \\ 0^T & N^T \end{array} \right] \left\{ \begin{array}{c} \tilde{\Phi}_r \\ \tilde{\Phi}_z \end{array} \right\} dS$$

Using conventional notation, this eq. is

$$\int [B]^T \{\tilde{\sigma}\} dV = \int [N]^T \{\tilde{\Phi}\} dS \quad (C)$$

$$\text{R.t. } \{\tilde{\sigma}\} = [E] \{\tilde{\epsilon}\} = [E][B] \{d\} \quad (D)$$

where, with $\{d\} = [u_1, u_2, \dots, w_1, w_2, \dots]^T$,

$$\{\tilde{\epsilon}\} = \left\{ \begin{array}{c} \tilde{\epsilon}_r \\ \tilde{\epsilon}_z \\ \tilde{\gamma}_{rz} \end{array} \right\} = \left[\begin{array}{cc} N_{,r} & 0 \\ 0 & N_{,z} \\ \frac{1}{r} N & 0 \\ N_{,z} & N_{,r} \end{array} \right] \{d\}$$

Eqs. (C) & (D) yield the standard result:

$$\int [B]^T [E] [B] dV \{d\} = \int [N]^T \{\tilde{\Phi}\} dS$$

5.6-1 Apply Eq. 5.6-9. Including all six d.o.f. of the two-el. model,

$$A \begin{bmatrix} -L/3E & -1/2 & -L/6E & 1/2 & 0 & 0 \\ -1/2 & 0 & -1/2 & 0 & 0 & 0 \\ -L/6E & -1/2 & -L/3E & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ u_1 \\ \sigma_2 \\ u_2 \\ \sigma_3 \\ u_3 \end{Bmatrix} +$$

$$A \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -L/3E & -1/2 & -L/6E & 1/2 \\ 0 & 0 & -1/2 & 0 & -1/2 & 0 \\ 0 & 0 & -L/6E & -1/2 & -L/3E & 1/2 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ u_1 \\ \sigma_2 \\ u_2 \\ \sigma_3 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} \text{load terms} \end{Bmatrix}$$

Combine. Discard rows & columns 2 and 5 to impose boundary conditions $u_1 = 0$ and $\sigma_3 = 0$. Include load terms $F_{q1} = cL/2$ and $F_{q2} = cL/2$ (as in Eq. 5.6-8). Thus

$$A \begin{bmatrix} -L/3E & -L/6E & 1/2 & 0 \\ -L/6E & -2L/3E & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ cL/2 \\ cL/2 \end{Bmatrix}$$

Solution is $\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} cL/A \\ cL/A \\ cL^2/AE \\ 5cL^2/3AE \end{Bmatrix}$ ✓
 ✓
 ✓
 ← exact is $\frac{3cL^2}{2AE}$

5.6-2

Governing eqs. are $v_{,xx} - \frac{M}{EI} = 0$ and $M_{,xx} - q = 0$

Assume, for element fields, $\tilde{v} = \tilde{N}\tilde{v}_e$ and $\tilde{M} = \tilde{N}\tilde{M}_e$

First eq.: $\int_0^L \tilde{N}^T (\tilde{v}_{,xx} - \frac{\tilde{M}}{EI}) dx = 0$. Integrate 1st term by parts:

$$\int_0^L \tilde{N}^T \tilde{v}_{,xx} dx = \left[\tilde{N}^T \tilde{v}_{,x} \right]_0^L - \int_0^L \tilde{N}_{,x}^T \tilde{v}_{,x} dx \quad \text{Hence 1st eq. becomes}$$

$$- \underbrace{\int_0^L \tilde{N}_{,x}^T \tilde{N}_{,x} dx}_{\tilde{H}_{12}} \tilde{v}_e - \underbrace{\int_0^L \frac{1}{EI} \tilde{N}^T \tilde{N} dx}_{\tilde{H}_{11}} \tilde{M}_e = - \left[\tilde{N}^T \tilde{v}_{,x} \right]_0^L$$

↑ Cancels upon assembly of elements

Second eq.: $\int_0^L \tilde{N}^T (\tilde{M}_{,xx} - q) dx = 0$. Integrate 1st term by parts:

$$\int_0^L \tilde{N}^T \tilde{M}_{,xx} dx = \left[\tilde{N}^T \tilde{M}_{,x} \right]_0^L - \int_0^L \tilde{N}_{,x}^T \tilde{M}_{,x} dx \quad \text{Hence 2nd eq. becomes}$$

$$- \underbrace{\int_0^L \tilde{N}_{,x}^T \tilde{N}_{,x} dx}_{\tilde{H}_{12}} \tilde{M}_e = \underbrace{\int_0^L \tilde{N}^T q dx}_{\tilde{r}_q} - \left[\tilde{N}^T \tilde{M}_{,x} \right]_0^L$$

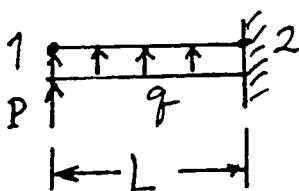
↑ Cancels upon assembly of els.

Put together: $\begin{bmatrix} \tilde{H}_{11} & \tilde{H}_{12} \\ \tilde{H}_{12} & 0 \end{bmatrix} \begin{Bmatrix} \tilde{M}_e \\ \tilde{v}_e \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\tilde{r}_q \end{Bmatrix}$ in which, if $[L\tilde{N}] = \begin{bmatrix} \frac{L-x}{L} & \frac{x}{L} \end{bmatrix}$,

$$[\tilde{H}_{11}] = \frac{L}{6EI} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad [\tilde{H}_{12}] = \frac{1}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

hence $\begin{bmatrix} L/3EI & L/6EI & 1/L & -1/L \\ L/6EI & L/3EI & -1/L & 1/L \\ 1/L & -1/L & 0 & 0 \\ -1/L & 1/L & 0 & 0 \end{bmatrix} \begin{Bmatrix} M_1 \\ M_2 \\ v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -r_{q1} \\ -r_{q2} \end{Bmatrix}$

Example: $M_1=0, v_2=0$, so



$$\begin{bmatrix} L/3EI & -1/L \\ -1/L & 0 \end{bmatrix} \begin{Bmatrix} M_2 \\ v_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P - qL/2 \end{Bmatrix}$$

Gives $M_2 = PL + \frac{qL^2}{2}$, $v_1 = \frac{PL^3}{3EI} + \frac{qL^4}{6EI}$

Should be $\frac{qL^4}{8EI}$; other terms in M_2 and v_1 are exact.

5.6-3

$$\Pi = \int \left(\sigma_x \frac{du}{dx} - \frac{\sigma_x^2}{2E} - \frac{q}{A} u \right) A dx$$

In an element, $\sigma_x = \tilde{N} \tilde{\sigma}_e$ hence $\sigma_x^T = \tilde{\sigma}_e^T \tilde{N}^T$
 $u = \tilde{N} \tilde{u}_e$ $u_{,x} = \tilde{N}_{,x} \tilde{u}_e$

$$\Pi = \int \left(\tilde{\sigma}_e^T \tilde{N}^T \tilde{N}_{,x} \tilde{u}_e - \frac{1}{2E} \tilde{\sigma}_e^T \tilde{N}^T \tilde{N} \tilde{\sigma}_e - \tilde{u}_e^T \tilde{N}^T \frac{q}{A} \right) A dx$$

$$\frac{\partial \Pi}{\partial \tilde{\sigma}_e} = \underbrace{\int \tilde{N}^T \tilde{N}_{,x} A dx}_{[k_{\sigma u}]} \tilde{u}_e - \underbrace{\int \tilde{N}^T \tilde{N} \frac{A}{E} dx}_{[k_{\sigma\sigma}]} \tilde{\sigma}_e$$

$$\frac{\partial \Pi}{\partial \tilde{u}_e} = \underbrace{\int \tilde{N}_{,x}^T \tilde{N} A dx}_{[k_{u\sigma}]} \tilde{\sigma}_e - \underbrace{\int \tilde{N}^T q dx}_{\{r_f\}}$$

Thus
$$\begin{bmatrix} -k_{\sigma\sigma} & k_{\sigma u} \\ k_{u\sigma} & 0 \end{bmatrix} \begin{Bmatrix} \tilde{\sigma}_e \\ \tilde{u}_e \end{Bmatrix} = \begin{Bmatrix} 0 \\ r_f \end{Bmatrix} \quad \checkmark$$

6.1-1

(a) Solve first of Eqs. 6.1-2 for the a_i :

2nd row gives $a_1 = x_2$, so 1st and 3rd rows become

$$x_1 - x_2 = -a_2 + a_3$$

$$x_3 - x_2 = a_2 + a_3$$

$$\text{from which } a_2 = \frac{-x_1 + x_3}{2}$$

$$a_3 = \frac{x_1 - 2x_2 + x_3}{2}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix}}_{[A]^{-1}} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

$$[N] = [1 \quad \xi \quad \xi^2][A]^{-1} = \begin{bmatrix} \frac{-\xi + \xi^2}{2} & 1 - \xi^2 & \frac{\xi + \xi^2}{2} \end{bmatrix}$$

(b) With ξ in place of x , Eq. 3.2-7 is

$$[N] = \begin{bmatrix} \frac{(\xi_2 - \xi)(\xi_3 - \xi)}{(\xi_2 - \xi_1)(\xi_3 - \xi_1)} & \frac{(\xi_1 - \xi)(\xi_3 - \xi)}{(\xi_1 - \xi_2)(\xi_3 - \xi_2)} & \frac{(\xi_1 - \xi)(\xi_2 - \xi)}{(\xi_1 - \xi_3)(\xi_2 - \xi_3)} \end{bmatrix}$$

Set $\xi_1 = -1$, $\xi_2 = 0$, $\xi_3 = 1$. Thus

$$\begin{aligned} [N] &= \begin{bmatrix} \frac{-\xi(1-\xi)}{(1)(2)} & \frac{(-1-\xi)(1-\xi)}{(-1)(1)} & \frac{(-1-\xi)(-\xi)}{(-2)(-1)} \end{bmatrix} \\ &= \begin{bmatrix} \frac{-\xi + \xi^2}{2} & 1 - \xi^2 & \frac{\xi + \xi^2}{2} \end{bmatrix} \end{aligned}$$

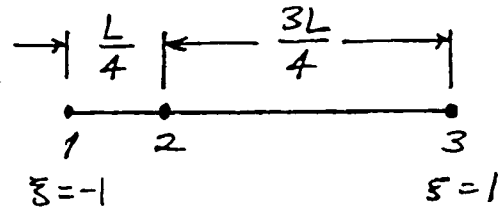
6.1-2

$$(a) \quad J = \begin{bmatrix} \frac{-1+2\xi}{2} & -2\xi & \frac{1+2\xi}{2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_1 + L/2 \\ x_1 + L \end{Bmatrix} = -2\xi \frac{L}{2} + \frac{1+2\xi}{2} L = \frac{L}{2}$$

(b) ϵ_x becomes infinite at node 1 if $J=0$ in Eq. 6.1-7 at $\xi=-1$

From Eq. 6.1-6, with $\xi=-1$,

$$0 = \begin{bmatrix} -\frac{3}{2} & 2 & -\frac{1}{2} \end{bmatrix} \begin{Bmatrix} 0 \\ x_2 \\ L \end{Bmatrix} = 2x_2 - \frac{L}{2} \quad \text{so } x_2 = \frac{L}{4}$$



6.1-3

From Problem 6.1-2a, $J = \frac{L}{2}$. Eq. 6.1-8 becomes

$$[\underline{k}] = \int_{-1}^1 [\underline{B}]^T [\underline{B}] AE \frac{L}{2} d\xi = \frac{2AE}{L} \int_{-1}^1 \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} d\xi$$

$$\text{where } a = \frac{1}{4}(-1+2\xi)^2 = \frac{1}{4}(1-4\xi+4\xi^2)$$

$$b = -\xi(-1+2\xi) = \xi-2\xi^2$$

$$c = \frac{1}{4}(-1+2\xi)(1+2\xi) = \frac{1}{4}(-1+4\xi^2)$$

$$d = 4\xi^2$$

$$e = -\xi(1+2\xi) = -\xi-2\xi^2$$

$$f = \frac{1}{4}(1+2\xi)^2 = \frac{1}{4}(1+4\xi+4\xi^2)$$

$$\int_{-1}^1 a d\xi = \frac{14}{4(3)}$$

$$\int_{-1}^1 b d\xi = -\frac{4}{3}$$

$$\int_{-1}^1 c d\xi = \frac{2}{4(3)}$$

$$\int_{-1}^1 d d\xi = \frac{8}{3}$$

$$\int_{-1}^1 e d\xi = -\frac{4}{3}$$

$$\int_{-1}^1 f d\xi = \frac{14}{4(3)}$$

$$[\underline{k}] = \frac{AE}{3L} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

6.1-4

$$x = [N] \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad u = [N] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}, \quad \text{where}$$

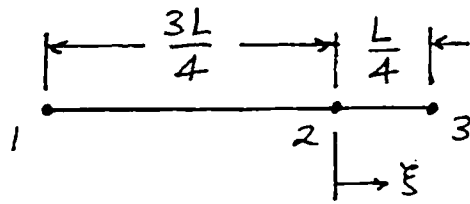
$$[N] = \frac{1}{2} \begin{bmatrix} 1-\xi & 1+\xi \end{bmatrix}, \quad J = \frac{dx}{d\xi} = \frac{x_2 - x_1}{2} = \frac{L}{2}$$

$$\epsilon_x = \frac{du}{dx} = \frac{du}{d\xi} \frac{d\xi}{dx} = \frac{1}{J} \frac{du}{d\xi} = \frac{2}{L} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \{d\} = \frac{1}{L} \underbrace{\begin{bmatrix} -1 & 1 \end{bmatrix}}_{[B]} \{d\}$$

$$[k] = \int_{-1}^1 [B]^T [B] AE J d\xi$$

$$[k] = \frac{AE}{2L} \int_{-1}^1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} d\xi = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

6.1-5



$$\text{Eq. 6.1-6: } J = \begin{bmatrix} \frac{1}{2}(-1+2\xi) & -2\xi & \frac{1}{2}(1+2\xi) \end{bmatrix} \begin{Bmatrix} 0 \\ 3L/4 \\ L \end{Bmatrix}$$

$$J = \frac{L}{2}(1-\xi)$$

If only u_3 is nonzero, Eqs. 6.1-5 and 6.1-7 yield

$$\epsilon_x = \frac{(1+2\xi)/2}{L(1-\xi)/2} u_3 = \frac{1+2\xi}{1-\xi} \frac{u_3}{L}$$

$$\text{Node 1: } \xi = -1, \epsilon_x = -\frac{1}{2} \frac{u_3}{L}$$

$$\text{Node 2: } \xi = 0, \epsilon_x = \frac{u_3}{L}$$

$$\text{Node 3: } \xi = 1, \epsilon_x = \frac{1}{0} \frac{u_3}{L} = \infty$$

6.2-1

(a) $x = [1, \xi, \eta, \xi\eta] \{a\}$. Subs. ξ & η coords. of nodes.

$$[x_1 \ x_2 \ x_3 \ x_4]^T = [\underline{A}] \{a\}, \quad [\underline{A}] = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

(b) $[N_1 \ N_2 \ N_3 \ N_4] =$

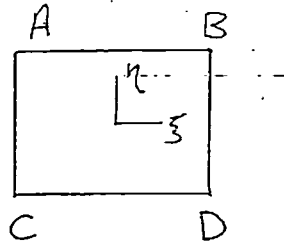
$[1 \ \xi \ \eta \ \xi\eta][\underline{A}]^{-1}$. By comparing this with Eqs. 6.2-3,

$$[\underline{A}]^{-1} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

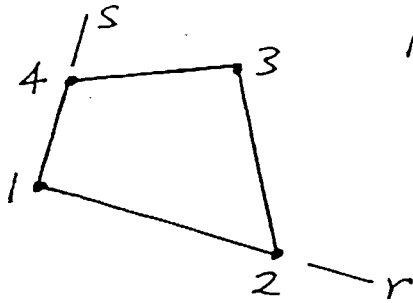
(c) Indeed $[\underline{A}][\underline{A}]^{-1} = [\underline{I}]$.

6.2-2

We answer by noting which of the N_i become unity when ξ & η define a corner coordinate.



6.2-3



By simple inspection and trial,

$$N_1 = (1-r)(1-s)$$

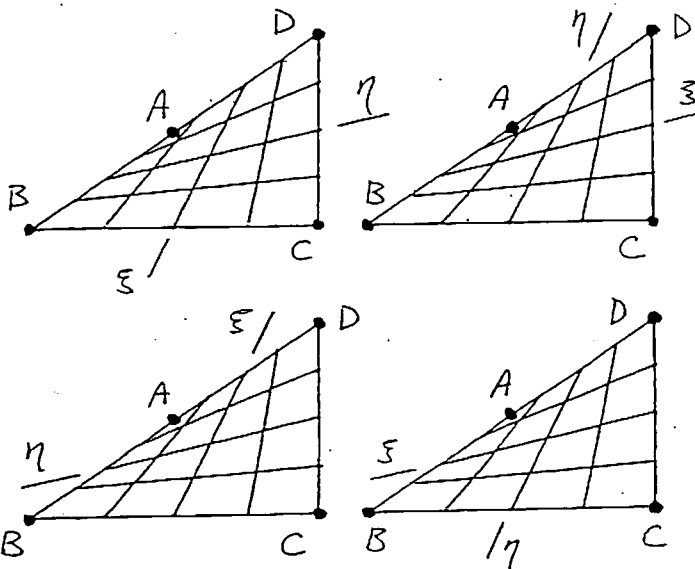
$$N_2 = r(1-s)$$

$$N_3 = rs$$

$$N_4 = (1-r)s$$

Each N_i is unity at node i and zero at node j , where $j \neq i$.

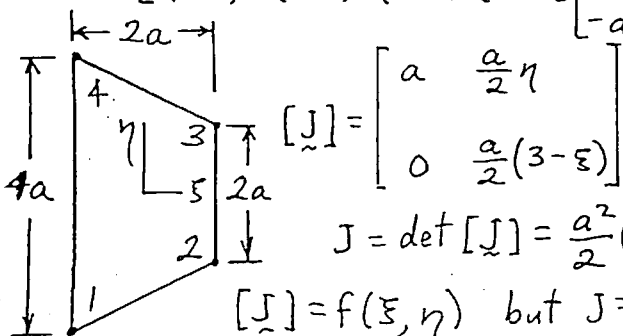
6.2-4



6.2-5

Apply Eq. 6.2-6 to the element shown.

$$[J] = \frac{1}{4} \begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1-\xi) & (1+\xi) & (1-\xi) \end{bmatrix} \begin{bmatrix} -a & -2a \\ a & -a \\ a & a \\ -a & 2a \end{bmatrix}$$



$$[J] = \begin{bmatrix} a & \frac{a}{2}\eta \\ 0 & \frac{a}{2}(3-\xi) \end{bmatrix}$$

$$J = \det [J] = \frac{a^2}{2}(3-\xi)$$

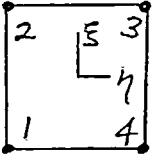
$$[J] = f(\xi, \eta) \text{ but } J = f(\xi)$$

6.2-6

From Eq. 6.2-6, let

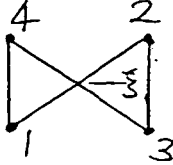
$$[\underline{D}_N] = \frac{1}{4} \begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi) \end{bmatrix}$$

Then

$$(a) [\underline{J}] = [\underline{D}_N] \begin{bmatrix} -1 & -1 \\ -1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$


$$J = |\underline{J}| = -1$$

Implies left-handed $\xi\eta$ axes.

$$(b) [\underline{J}] = [\underline{D}_N] \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\eta \\ 0 & -\xi \end{bmatrix}$$


$$J = |\underline{J}| = -\xi$$

Implies "bow-tie" element.

6.2-7

Use Eq. 6.2-6. Define the 2 by 8 matrix in Eq. 6.2-6 as $[D_N]$ and write it in the form

$$[D_N] = \frac{1}{4} \begin{bmatrix} -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} \eta & -\eta & \eta & -\eta \\ \xi & -\xi & \xi & -\xi \end{bmatrix}$$

$$(a) [D_N] = \begin{bmatrix} -3 & -2 \\ 3 & -2 \\ -3 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, J=6$$

$$(b) [D_N] = \begin{bmatrix} 3 & -2 \\ 3 & 2 \\ -3 & -2 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix}, J=6$$

$$(c) [D_N] = \begin{bmatrix} 0 & -2 \\ 3 & 0 \\ 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3/2 & -\eta \\ 0 & 1-\xi \end{bmatrix}, J = \frac{3}{2}(1-\xi)$$

$$(d) [D_N] = \begin{bmatrix} -3 & -2 \\ 0 & -2 \\ 3 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3/2 & 0 \\ 3/2 & 2 \end{bmatrix}, J=3$$

Area ratios $\begin{cases} A_{e1} \div A_{2x2} \\ J \end{cases}$ (a) (b) (c) (d)
 6 6 6 3
 6 6 3/2 3
 6 6 3/2(1-ξ) 3
 J is ratio $\frac{dx dy}{d\xi d\eta}$ & is unity if ξ=x & η=y, which is the case of 2x2 square.

6.3-1

$$\text{exact } I = \int_{-1}^1 \phi dx = 2a_1 + \frac{2}{3}a_3$$

Let W = weights, $\pm p$ = location.

$$W(a_1 - a_2 p + a_3 p^2 - a_4 p^3) + W(a_1 + a_2 p + a_3 p^2 + a_4 p^3) \\ = 2a_1 + \frac{2}{3}a_3$$

Reduces to

$$W a_1 + W a_3 p^2 = a_1 + \frac{1}{3} a_3$$

Must be true for any a_1 & a_3 , so

$$a_1 (W - 1) = 0 \quad (a)$$

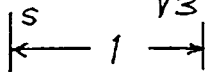
$$a_3 (W p^2 - \frac{1}{3}) = 0 \quad (b)$$

(a) yields $W = 1$, hence (b) yields

$$p = \pm \frac{1}{\sqrt{3}} = \pm 0.57735 \dots$$

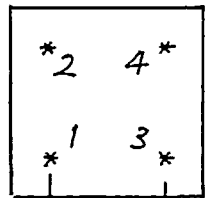
6.3-2

$$b = \frac{0.5}{\sqrt{3}}, a = 0.5 - b, c = 0.5 + b$$



$$a = 0.21132\dots$$

$$c = 0.78868\dots$$



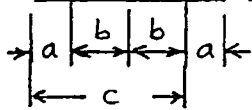
point	r_i	s_i
-------	-------	-------

$$1 \quad a \quad a$$

$$2 \quad a \quad c$$

$$3 \quad c \quad a$$

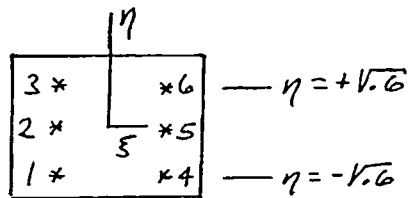
$$4 \quad c \quad c$$



Weights: $\frac{1}{2}$ each, since

$$\int_0^1 \int_0^1 dr ds = \sum_{i=1}^2 \sum_{j=1}^2 W_i W_j = 1$$

6.3-3



$$\xi = -\frac{\sqrt{3}}{3} \rightarrow \quad \leftarrow \xi = +\frac{\sqrt{3}}{3}$$

$$I = (1) \left(\frac{5}{9} \phi_1 + \frac{8}{9} \phi_2 + \frac{5}{9} \phi_3 \right) + (1) \left(\frac{5}{9} \phi_4 + \frac{8}{9} \phi_5 + \frac{5}{9} \phi_6 \right)$$

$$I = \frac{5}{9} (\phi_1 + \phi_3 + \phi_4 + \phi_6) + \frac{8}{9} (\phi_2 + \phi_5)$$

6.3-4

8 points nearest corners

$$\frac{W_i W_j W_k}{\left(\frac{5}{9}\right)^3}$$

6 points nearest middle of faces

$$\left(\frac{5}{9}\right)\left(\frac{8}{9}\right)^2$$

12 points nearest middle of edges

$$\left(\frac{5}{9}\right)^2\left(\frac{8}{9}\right)$$

1 point at center ($\xi = \eta = \zeta = 0$)

$$\left(\frac{8}{9}\right)^3$$

27 points ($= 3 \times 3 \times 3$)

Sum of weight products:

$$8\left(\frac{5}{9}\right)^3 + 6\left(\frac{5}{9}\right)\left(\frac{8}{9}\right)^2 + 12\left(\frac{5}{9}\right)^2\left(\frac{8}{9}\right) + \left(\frac{8}{9}\right)^3 = 8 \quad \checkmark$$

6.3-5

$$I_{\text{exact}} = \int_0^{12} y \, dx = 6 \frac{2+4}{2} + 6(4) = 18 + 24 = 42$$

$$x = 6 + 6\xi, \quad J = \frac{dx}{d\xi} = 6$$

$$I_1 = 6 [2(4)] = 48$$

error
+14.3%

$$I_2 = 6 \left[\left(4 - 2\frac{1}{\sqrt{3}}\right) + 4 \right] = 41.072$$

-2.2%

$$I_3 = 6 \left[\left(4 - 2\sqrt{0.6}\right) \frac{5}{9} + 4\left(\frac{8}{9}\right) + 4\left(\frac{5}{9}\right) \right] = 42.836$$

+2.0%

Convergence is not monotonic

6.3-6

$$(a) \text{ Exact: } \int_{-1}^1 (\xi^2 + \xi^3) d\xi = \frac{2}{3} = I$$

1 pt. $I_1 = 2(0+0) = 0$ 100% low

2 pts. Let $a = \sqrt{3}/3$, then

$$I_2 = (a^2 - a^3) + (a^2 + a^3) = 2a^2 = \frac{2}{3}$$

3 pts. Let $b = \sqrt{0.6}$, then exact

$$I_3 = \frac{5}{9}(b^2 - b^3) + \frac{8}{9}(0) + \frac{5}{9}(b^2 + b^3)$$

$$I_3 = \frac{10}{9}b^2 = \frac{2}{3} \text{ exact}$$

$$(b) \int_{-1}^1 \cos 1.5\xi d\xi = \frac{\sin 1.5\xi}{1.5} \Big|_{-1}^1 = 1.3300 = I$$

1 pt. $I_1 = 2(1) = 2$ +50.4%

2 pts. $I_2 = \cos\left(-\frac{1.5}{\sqrt{3}}\right) + \cos\left(\frac{1.5}{\sqrt{3}}\right) = 1.2957$
-2.58%

3 pts. $I_3 = \frac{5}{9} \cos(-1.5\sqrt{0.6}) + \frac{8}{9} \cos(0)$
 $+ \frac{5}{9} \cos(1.5\sqrt{0.6}) = 1.3307$
+0.05%

$$(c) \int_{-1}^1 \frac{1-\xi}{2+\xi} d\xi = \int_{-1}^1 \frac{d\xi}{2+\xi} - \int_{-1}^1 \frac{\xi d\xi}{2+\xi}$$

$$= \ln(2+\xi) \Big|_{-1}^1 - \left[2+\xi - 2\ln(2+\xi) \right] \Big|_{-1}^1$$

$$= \ln 3 - 2 + 2\ln 3 = 3\ln 3 - 2 = 1.2958 = I$$

1 pt. $I_1 = 2 \frac{1}{2} = 1$ -22.8%

2 pts. $I_2 = \frac{1 + \frac{1}{\sqrt{3}}}{2 - \frac{1}{\sqrt{3}}} + \frac{1 - \frac{1}{\sqrt{3}}}{2 + \frac{1}{\sqrt{3}}} = 1.2727$
-1.8%

3 pts. $I_3 = \frac{5}{9} \frac{1 + \sqrt{0.6}}{2 - \sqrt{0.6}} + \frac{8}{9} \frac{1}{2} + \frac{5}{9} \frac{1 - \sqrt{0.6}}{2 + \sqrt{0.6}}$
 $= 1.2941$ -0.13%

$$(d) x = \left[\frac{1-\xi}{2} \quad \frac{1+\xi}{2} \right] \left\{ \frac{1}{7} \right\} = \frac{8+6\xi}{2} = 4+3\xi$$

$$I = \int_1^7 \frac{dx}{x} = \ln 7 = 1.94591 = \int_{-1}^1 \frac{3 d\xi}{4+3\xi}$$

$I_1 = 2 \frac{3}{4} = 1.5$ -22.9%

$I_2 = \frac{3}{4-\sqrt{3}} + \frac{3}{4+\sqrt{3}} = 1.84615$ -5.1%

$I_3 = \frac{5}{9} \left(\frac{3}{4-3\sqrt{0.6}} + \frac{3}{4+3\sqrt{0.6}} \right) + \frac{8}{9} \frac{3}{4} = 1.92453$ -1.1%

6.3-7

$$\int_{-1}^1 (3 + \xi^2) d\xi = \left(3\xi + \frac{\xi^2}{3} \right) \Big|_{-1}^1 = \frac{20}{3}$$

$$I = \frac{20}{3} \int_{-1}^1 \frac{d\eta}{2 + \eta^2} = \frac{20}{3} \frac{1}{\sqrt{2}} \arctan \frac{\eta}{\sqrt{2}} \Big|_{-1}^1$$

$$I = 5.8028$$

$$1 \text{ pt. } I_1 = (2)(2) \left(\frac{3}{2} \right) = 6 \quad +3.4\%$$

$$2 \text{ pts. } I_2 = 4 \frac{3 + \frac{1}{3}}{2 + \frac{1}{3}} = 5.7143 \quad -1.53\%$$

$$3 \text{ pts. } I_3 = 4 \left(\frac{25}{81} \frac{3+0.6}{2+0.6} \right) + 2 \left(\frac{40}{81} \frac{3+0.6}{2} \right) \\ + 2 \left(\frac{40}{81} \frac{3}{2+0.6} \right) + \frac{64}{81} \frac{3}{2} = 5.8120 \quad +0.16\%$$

6.3-8

$$\begin{array}{c} 1 \xrightarrow{A, E} 2 \\ \leftarrow L \rightarrow \\ \xi = -1 \quad \xi = +1 \end{array} \quad u = \begin{bmatrix} \frac{1-\xi}{2} & \frac{1+\xi}{2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$J = L/2$

$$\epsilon_x = \frac{1}{J} \frac{du}{d\xi} = \frac{L}{2} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{L} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$[k] = \int_{-1}^1 \begin{bmatrix} B \\ \tilde{B} \end{bmatrix}^T \begin{bmatrix} B \\ \tilde{B} \end{bmatrix} A E J d\xi = \frac{E}{2L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_{-1}^1 A d\xi$$

(a) A const.; $\int_{-1}^1 A d\xi = A \int_{-1}^1 d\xi = A(1+1) = 2A$

$$[k] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{aligned} (b) \int_{-1}^1 A d\xi &= \int_{-1}^1 \left(\frac{1-\xi}{2} A_1 + \frac{1+\xi}{2} A_2 \right) d\xi \\ &= (1) \left(\frac{1+1/\sqrt{3}}{2} A_1 + \frac{1-1/\sqrt{3}}{2} A_2 \right) + (1) \left(\frac{1-1/\sqrt{3}}{2} A_1 + \frac{1+1/\sqrt{3}}{2} A_2 \right) \left. \begin{array}{l} 2\text{-pt.} \\ \text{Gauss} \\ \text{rule} \end{array} \right\} \\ &= A_1 + A_2, \text{ so } [k] = \frac{A_1 + A_2}{2} \frac{E}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{aligned}$$

(c) Evaluate integral of (b) at $\xi=0$ with weight 2. Thus

$$\int_{-1}^1 A d\xi = 2 \left(\frac{A_1}{2} + \frac{A_2}{2} \right) = A_1 + A_2$$

Same result as in part (b).

6.3-9

From Eq. 6.1-7, with $J = L/2$,

$$\epsilon_x = \frac{1}{L} \underbrace{\begin{bmatrix} (-1+2\xi) & -4\xi & (1+2\xi) \end{bmatrix}}_{[B_1]} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{1}{L} [B_1] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$[k] = \int_{-1}^1 AE [B_1]^T [B_1] J d\xi = AE \frac{L}{2} \frac{1}{L^2} \int_{-1}^1 [B_1]^T [B_1] d\xi$$

$$[k] = \frac{AE}{2L} \underbrace{[B_1]^T [B_1]}_{\text{at } \xi = -\frac{1}{\sqrt{3}}} + \frac{AE}{2L} \underbrace{[B_1]^T [B_1]}_{\text{at } \xi = +\frac{1}{\sqrt{3}}}$$

$$[k] = \frac{AE}{2L} \begin{Bmatrix} -2.1547 \\ 2.3094 \\ -0.1547 \end{Bmatrix} [---] + \frac{AE}{2L} \begin{Bmatrix} 0.1547 \\ -2.3094 \\ 2.1547 \end{Bmatrix} [---]$$

$$[k] = \frac{AE}{2L} \begin{bmatrix} 4.6427 & -4.9761 & 0.3333 \\ & 5.3333 & -0.3573 \\ \text{symm.} & & 0.0239 \end{bmatrix} + \frac{AE}{2L} \begin{bmatrix} 0.0239 & -0.3573 & 0.3333 \\ & 5.3333 & -4.9761 \\ \text{symm.} & & 4.6427 \end{bmatrix}$$

$$[k] = \frac{AE}{3L} \begin{bmatrix} 4\frac{2}{3} & -5\frac{1}{3} & \frac{2}{3} \\ & 10\frac{2}{3} & -5\frac{1}{3} \\ \text{symm.} & & 4\frac{2}{3} \end{bmatrix} = \frac{AE}{3L} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

6.3-10

$$[B] = [B_1 \ B_2 \ -B_1 \ B_4] \quad \text{where, with } x = \frac{L}{2}(1+\xi),$$

$$B_1 = -\frac{6}{L^2} + \frac{12x}{L^3} = -\frac{6}{L} + \frac{12}{L^3} \frac{L}{2}(1+\xi) = \frac{6\xi}{L^2}$$

$$B_2 = -\frac{4}{L} + \frac{6x}{L^2} = -\frac{4}{L} + \frac{6}{L^2} \frac{L}{2}(1+\xi) = \frac{1}{L}(-1+3\xi)$$

$$B_4 = -\frac{2}{L} + \frac{6x}{L^2} = -\frac{2}{L} + \frac{6}{L^2} \frac{L}{2}(1+\xi) = \frac{1}{L}(1+3\xi)$$

$$k_{11} = EI \int_{-1}^1 B_1^2 \frac{L}{2} d\xi = \frac{EIL}{2} \int_{-1}^1 \frac{36\xi^2}{L^4} d\xi = \frac{18EI}{L^3} \left(\frac{1}{3} + \frac{1}{3} \right) = \frac{12EI}{L^3}$$

$$k_{12} = EI \int_{-1}^1 B_1 B_2 \frac{L}{2} d\xi = \frac{3EI}{L^2} \int_{-1}^1 (-\xi + 3\xi^2) d\xi = \frac{3EI}{L^2} \left[\left(-\frac{1}{\sqrt{3}} + \frac{3}{3} \right) + \left(-\frac{1}{\sqrt{3}} + \frac{3}{3} \right) \right] = \frac{6EI}{L^2}$$

$$k_{13} = -k_{11} \text{ by inspection}$$

$$k_{14} = EI \int_{-1}^1 B_1 B_4 \frac{L}{2} d\xi = \frac{3EI}{L^2} \int_{-1}^1 (\xi + 3\xi^2) d\xi = \frac{3EI}{L^2} \left[\left(-\frac{1}{\sqrt{3}} + \frac{3}{3} \right) + \left(\frac{1}{\sqrt{3}} + \frac{3}{3} \right) \right] = \frac{6EI}{L^2}$$

$$k_{22} = EI \int_{-1}^1 B_2^2 \frac{L}{2} d\xi = \frac{EI}{2L} \int_{-1}^1 (-1+3\xi)^2 d\xi = \frac{EI}{2L} \left[\left(-1 - \frac{3}{\sqrt{3}} \right)^2 + \left(-1 + \frac{3}{\sqrt{3}} \right)^2 \right]$$

$$= \frac{EI}{2L} [7.464 + 0.536] = \frac{4EI}{L}$$

$$k_{23} = -k_{12} \text{ by inspection}$$

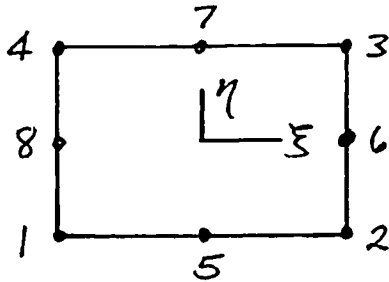
$$k_{24} = \frac{EI}{2L} \int_{-1}^1 B_2 B_4 \frac{L}{2} d\xi = \frac{EI}{2L} \int_{-1}^1 (-1+9\xi^2) d\xi = \frac{EI}{2L} \left[\left(-1 + \frac{9}{3} \right) + \left(-1 + \frac{9}{3} \right) \right] = \frac{2EI}{L}$$

$$k_{33} = k_{11} \text{ by inspection}$$

$$k_{34} = -k_{14} \text{ by inspection}$$

$$k_{44} = EI \int_{-1}^1 B_4^2 \frac{L}{2} d\xi = \frac{EI}{2L} \left[\left(1 - \frac{6}{\sqrt{3}} + \frac{9}{3} \right) + \left(1 + \frac{6}{\sqrt{3}} + \frac{9}{3} \right) \right] = \frac{4EI}{L}$$

6.4-1



Left edge

$$N_4 = \frac{1}{2}(1+\eta) - \frac{1}{2}(1-\eta^2) = \frac{\eta+\eta^2}{2}$$

$$N_8 = 1-\eta^2$$

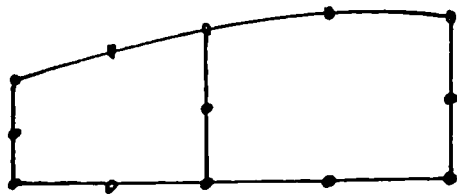
$$N_1 = \frac{1}{2}(1-\eta) - \frac{1}{2}(1-\eta^2) = \frac{-\eta+\eta^2}{2}$$

Right edge

$$N_3 = \frac{1}{2}(1+\eta) - \frac{1}{2}(1-\eta^2) = \frac{\eta+\eta^2}{2}$$

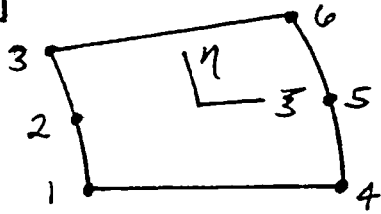
$$N_6 = 1-\eta^2$$

$$N_2 = \frac{1}{2}(1-\eta) - \frac{1}{2}(1-\eta^2) = \frac{-\eta+\eta^2}{2}$$



Hence on a common edge such as this, the field quantity is defined by the same three nodal d.o.f. and uses the same N_i whether viewed from the left element or the right, so the identical curve $\phi = \phi(\eta)$ is produced.

6.4-2



Can use Eq. 6.1-4, but apply to sides 1-2-3 and 4-5-6 :

$$\text{On } \xi = -1, \quad \phi = -\frac{\eta + \eta^2}{2} \phi_1 + (1 - \eta^2) \phi_2 + \frac{\eta + \eta^2}{2} \phi_3$$

$$\text{On } \xi = +1, \quad \phi = -\frac{\eta + \eta^2}{2} \phi_4 + (1 - \eta^2) \phi_5 + \frac{\eta + \eta^2}{2} \phi_6$$

Sweep using shape function $\frac{1-\xi}{2}$ for left edge, $\frac{1+\xi}{2}$ for right.
Thus for the six-node element,

$$N_1 = \frac{1-\xi}{2} \left(-\frac{\eta + \eta^2}{2} \right)$$

$$N_4 = \frac{1+\xi}{2} \left(-\frac{\eta + \eta^2}{2} \right)$$

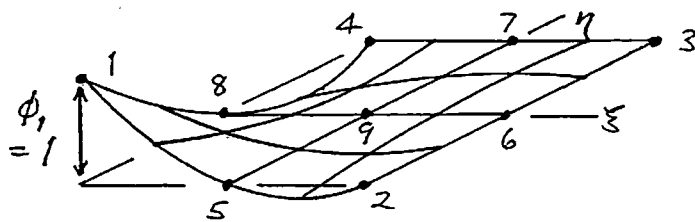
$$N_2 = \frac{1-\xi}{2} (1 - \eta^2)$$

$$N_5 = \frac{1+\xi}{2} (1 - \eta^2)$$

$$N_3 = \frac{1-\xi}{2} \left(\frac{\eta + \eta^2}{2} \right)$$

$$N_6 = \frac{1+\xi}{2} \left(\frac{\eta + \eta^2}{2} \right)$$

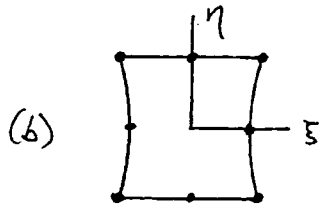
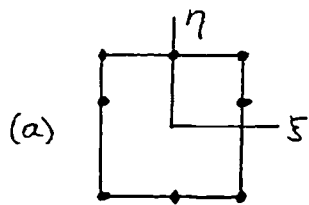
6.4-3



$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) - \frac{1}{4}(1-\xi^2)(1-\eta) - \frac{1}{4}(1-\xi)(1-\eta^2) + \frac{1}{4}(1-\xi^2)(1-\eta^2)$$

ξ	η	$N_1/64$
$-1/2$	$-1/2$	$36 - 18 - 18 + 9 = 9 \uparrow$
$1/2$	$-1/2$	$12 - 18 - 6 + 9 = -3 \downarrow$
$1/2$	$1/2$	$4 - 6 - 6 + 9 = 1 \uparrow$
$-1/2$	$1/2$	$12 - 6 - 18 + 9 = -3 \downarrow$

6.4-4



6.4-5

$$\text{On } \xi = -1, \phi = \eta \phi_4 + (1-\eta) \phi_8 \quad (A)$$

$$\text{On } \eta = -1, \phi = \xi \phi_2 + (1-\xi) \phi_5 \quad (B)$$

At node 1, $\xi = \eta = -1$, (A) & (B) give

$$\phi_1 = -\phi_4 - 2\phi_8 = -\phi_2 - 2\phi_5 \quad (C)$$

But ϕ_2, ϕ_4, ϕ_5 & ϕ_8 are independent, which contradicts (C).

Also, if N_1 of Table 6.6-1 omitted, ϕ along $\xi = -1$ is quadratic in η (from N_8), which contradicts (A). Similar for ϕ along $\eta = -1$ & (C).

6.4-6

For a beam that extends from $-\xi$ to $+\xi$ in natural coordinates, consider

$$v = a_1 (1 + \cos \pi \xi)$$

where a_1 is a generalized d.o.f. Hence

$$\frac{dv}{dx} = \frac{2}{L} \frac{dv}{d\xi} = -a_1 \frac{2}{L} (\pi \sin \pi \xi)$$

At ends $\pm L/2$, v and dv/dx both vanish; OK.

6.6-1

Consider Eqs. 3.6-10, which are for a Q4 element in pure bending:

$$\epsilon_x = -\frac{\theta_{el} y}{2a} \quad \epsilon_y = 0 \quad \gamma_{xy} = -\frac{\theta_{el} x}{2a}$$

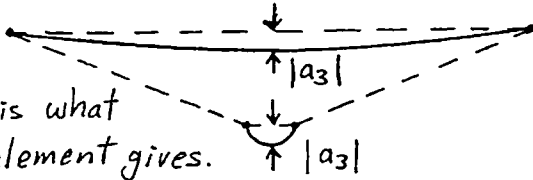
These equations pertain to a four-node element in which bending dominates and internal d.o.f. are omitted. If nodal d.o.f. (and hence θ_{el}) are rather accurate, then ϵ_x is rather accurate but ϵ_y and γ_{xy} are not. Thus for stresses:

$$\sigma_x = \frac{E}{1-\nu^2} (\epsilon_x + \nu \epsilon_y) \quad \text{some error}$$

$$\sigma_y = \frac{E}{1-\nu^2} (\epsilon_y + \nu \epsilon_x) \quad \text{larger error}$$

$$\tau_{xy} = G \gamma_{xy} \quad \text{very large error}$$

6.6-2



This is what
the element gives.

But, for a correct model of pure bending, both arcs should have the same center (the center of curv. of the beam).

6.7-1

$$\begin{bmatrix} 12 & -6 & 0 \\ -6 & 12 & -6 \\ 0 & -6 & 6 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 24 \\ 24 \\ 0 \end{Bmatrix}, \text{ or } \begin{bmatrix} 6 & -6 & 0 \\ -6 & 12 & -6 \\ 0 & -6 & 12 \end{bmatrix} \begin{Bmatrix} u_4 \\ u_3 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 24 \\ 24 \end{Bmatrix}$$

$$[k_{cc}] = \begin{bmatrix} 12 & -6 \\ -6 & 12 \end{bmatrix}, [k_{cc}]^{-1} = \frac{1}{18} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[k]_{\text{cond.}} = 6 - \begin{bmatrix} -6 & 0 \end{bmatrix} \frac{1}{18} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} -6 \\ 0 \end{Bmatrix} = 6 - 4 = 2$$

$$\{r\}_{\text{cond.}} = 0 - \begin{bmatrix} -6 & 0 \end{bmatrix} \frac{1}{18} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} 24 \\ 24 \end{Bmatrix} = \frac{6}{18} 72 = 24$$

$$\text{i.e. } 2u_4 = 24 \text{ so } u_4 = 12$$

Recover $\{d_c\}$:

$$\begin{Bmatrix} u_3 \\ u_2 \end{Bmatrix} = -\frac{1}{18} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \left(\begin{Bmatrix} -6 \\ 0 \end{Bmatrix} 12 - \begin{Bmatrix} 24 \\ 24 \end{Bmatrix} \right) = \begin{Bmatrix} 12 \\ 8 \end{Bmatrix}$$

Check: (order of d.o.f. here as originally written)

$$\begin{bmatrix} 12 & -6 & 0 \\ -6 & 12 & -6 \\ 0 & -6 & 6 \end{bmatrix} \begin{Bmatrix} 8 \\ 12 \\ 12 \end{Bmatrix} = \begin{Bmatrix} 24 \\ 24 \\ 0 \end{Bmatrix} \quad \checkmark$$

6.7-2

(a) Fill in column 1 of $[k]$ by symmetry: $[k] = \frac{AE}{3L} \begin{bmatrix} 7 & -8 & 1 \\ -8 & & \\ 1 & & \end{bmatrix}$

D.o.f. u_1 and u_3 create the same nodal forces; using this, and symmetry to fill in the last row,

$$[k] = \frac{AE}{3L} \begin{bmatrix} 7 & -8 & 1 \\ -8 & & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

Finally, there is no force at node 2 if $u_1 = u_2 = u_3$, so

$$[k] = \frac{AE}{3L} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

(b) To suit explanation in Section 6.7, reorder d.o.f.

$$[k] = \frac{AE}{3L} \begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix} \begin{matrix} u_1 \\ u_3 \\ u_2 \end{matrix} \quad \text{To condense } u_2, \text{ apply Eq. 6.7-3:}$$

$$[k_{cc}] = \frac{AE}{3L} 16, \quad [k_{cc}]^{-1} = \frac{3L}{16AE}$$

$$[k_{rc}] = \frac{AE}{3L} \begin{Bmatrix} -8 \\ -8 \end{Bmatrix}$$

$$[k_{cond}] = \frac{AE}{3L} \begin{bmatrix} 7 & 1 \\ 1 & 7 \end{bmatrix} - \frac{AE}{3L} \begin{Bmatrix} -8 \\ -8 \end{Bmatrix} \frac{3L}{16AE} \frac{AE}{3L} \begin{bmatrix} -8 & -8 \end{bmatrix}$$

$$= \frac{AE}{3L} \begin{bmatrix} 7 & 1 \\ 1 & 7 \end{bmatrix} - \frac{AE}{3L} \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} u_1 \\ u_3 \end{matrix}$$

(c) With d.o.f. in the order u_1, u_3, u_2 , the load vector is $\{r_e\} = \frac{qL}{6} \begin{Bmatrix} 1 \\ 1 \\ 4 \end{Bmatrix}$

$$\{r_{cond}\} = \frac{qL}{6} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} - \frac{AE}{3L} \begin{Bmatrix} -8 \\ -8 \end{Bmatrix} \frac{3L}{16AE} \frac{4qL}{6} = \frac{qL}{6} \begin{Bmatrix} 1+2 \\ 1+2 \end{Bmatrix} = \frac{qL}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\text{Condensed system: } \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_3 \end{Bmatrix} = \frac{qL}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \text{Gives } u_3 = \frac{qL^2}{2AE} \text{ if } u_1 = 0$$

Then from Eq. 6.7-2:

$$u_2 = \{d_c\} = -\frac{3L}{16AE} \left(\frac{AE}{3L} \begin{bmatrix} -8 & -8 \end{bmatrix} \begin{Bmatrix} 0 \\ qL^2/2AE \end{Bmatrix} - \frac{4qL}{6} \right) = \frac{3qL^2}{8AE}$$

Check nodal loads using computed d.o.f.:

$$\frac{AE}{3L} \begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix} \begin{Bmatrix} 0 \\ qL^2/2AE \\ 3qL^2/8AE \end{Bmatrix} = \frac{qL}{3} \begin{Bmatrix} \frac{1}{2} - 3 \\ \frac{7}{2} - 3 \\ -4 + 6 \end{Bmatrix} = qL \begin{Bmatrix} -5/6 \\ 1/6 \\ 2/3 \end{Bmatrix} \quad \text{Satisfies equil.}$$

6.7-3

$$[k] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{matrix} v_1 \\ \theta_{z1} \\ v_2 \\ \theta_{z2} \end{matrix}$$

Condense θ_{z2} : apply Eq. 6.7-3

$$\begin{aligned} [k_{cond}] &= \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 \\ 6L & 4L^2 & -6L \\ -12 & -6L & 12 \end{bmatrix} - \frac{EI}{L^3} \begin{Bmatrix} 6L \\ 2L^2 \\ -6L \end{Bmatrix} \frac{L}{4EI} \frac{EI}{L^3} [6L \ 2L^2 \ -6L] \\ &= \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 \\ 6L & 4L^2 & -6L \\ -12 & -6L & 12 \end{bmatrix} - \frac{EI}{L^3} \begin{bmatrix} 9 & 3L & -9 \\ 3L & L^2 & -3L \\ -9 & -3L & 9 \end{bmatrix} \\ &= \frac{EI}{L^3} \begin{bmatrix} 3 & 3L & -3 \\ 3L & 3L^2 & -3L \\ -3 & -3L & 3 \end{bmatrix} \\ &= \frac{3EI}{L^3} \begin{bmatrix} 1 & L & -1 \\ L & L^2 & -L \\ -1 & -L & 1 \end{bmatrix} \end{aligned}$$

6.7-4

Let $[k_I] = \begin{bmatrix} [k] & [0] \\ [0] & [0] \end{bmatrix}$ in which $[k]$ appears in Eq. 2.3-6

Let $[k_{II}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_1 & 0 & 0 & 0 & -k_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_2 & 0 & -k_2 \\ 0 & 0 & -k_1 & 0 & 0 & 0 & k_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -k_2 & 0 & k_2 \end{bmatrix}$

Both $[k_I]$ and $[k_{II}]$ operate on the nodal d.o.f. $\{d\} = \begin{Bmatrix} u_1 \\ v_1 \\ \theta_{21} \\ u_2 \\ v_2 \\ \theta_{22} \\ \beta_{21} \\ \beta_{22} \end{Bmatrix}$

Form $[k_I] + [k_{II}]$, condense out d.o.f.

θ_{21} and θ_{22} , discard the rows and columns

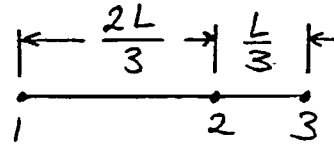
that correspond to θ_{21} and θ_{22} ; what remains

is a 6 by 6 matrix that operates on nodal d.o.f. $\{d_c\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \beta_{21} \\ \beta_{22} \end{Bmatrix}$

(Rearrange as may be convenient or necessary.)

6.8-1

$$e_x = \frac{1}{J} \begin{bmatrix} -1+2\xi & -2\xi & \frac{1+2\xi}{2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$



$$J = \begin{bmatrix} -1+2\xi & -2\xi & \frac{1+2\xi}{2} \end{bmatrix} \begin{Bmatrix} 0 \\ 2L/3 \\ L \end{Bmatrix} \quad \text{For } \xi=0, J=J_0 = \frac{L}{2}$$

$$\text{For } \xi=0, \quad e_x = \underbrace{\frac{2}{L} \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}}_{[B_0]} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$[k] = \int_{-1}^1 [B]^T [B] AE J d\xi$$

For one-point quadrature, with weight factor $W=2$,

$$[k] = W [B_0]^T [B_0] AE J_0 = 2AE \frac{L}{2} \left(\frac{2}{L}\right)^2 \begin{Bmatrix} -1/2 \\ 0 \\ 1/2 \end{Bmatrix} \begin{bmatrix} -1/2 & 0 & 1/2 \end{bmatrix}$$

$$[k] = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

6.8-2

$$V = \int_{-1}^1 \int_{-1}^1 J t \, d\xi \, d\eta$$

(a) From Eq. 6.2-6, powers of ξ & η in $[J]$ go to $\begin{bmatrix} \eta' & \eta' \\ \xi' & \xi' \end{bmatrix}$. Also, t is bilinear.

Hence Jt contains terms to $\xi^2\eta$, $\xi\eta^2$, and $\xi^2\eta^2$. Second powers: need 2×2 .

(b) In similar fashion, powers in $[J]$ and t go to $\begin{bmatrix} \eta^2 & \eta^2 \\ \xi^2 & \xi^2 \end{bmatrix}$ and $\xi\eta^2$, $\xi^2\eta$ in t .

Jt contains ξ^4 & η^4 . Need 3×3 rule: $(2 \cdot 3 - 1) = 5 > 4$.

6.8-3

$$V = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 J d\xi d\eta d\zeta.$$

Eq. 6.5-2 shows
that in $[J]$ we
will find powers of
 ξ, η, ζ up to: $\rightarrow \begin{bmatrix} \eta\zeta & \eta\zeta & \eta\zeta \\ \xi\zeta & \xi\zeta & \xi\zeta \\ \xi\eta & \xi\eta & \xi\eta \end{bmatrix}$
Determinant(J) contains $\xi^2, \eta^2, \& \zeta^2$.
Need $2 \times 2 \times 2$ rule: $(2 \cdot 2 - 1) = 3 > 2$.

6.8-4

Jacobian J is constant. Look for highest powers of ξ and η in the product $[\underline{B}]^T[\underline{B}]t$, where $t = \sum N_i t_i$

(a) Q4 element:

From Eqs. 6.2-3, t displays ξ^1 and η^1

From Eq. 6.2-11, $[\underline{B}]$ displays ξ^1 and η^1 (J is constant)

Hence $[\underline{B}]^T[\underline{B}]t$ displays ξ^3 and η^3 ; need a 2 by 2 rule

(b) Q8 element:

From Eqs. 6.4-2, t displays ξ^2 and η^2

From Table 6.4-1, $[\underline{B}]$ will contain ξ^2 and η^2 (J is constant)

Hence $[\underline{B}]^T[\underline{B}]t$ displays ξ^6 and η^6 ; need a 4 by 4 rule

6.8-5

For rectangular elements the 2 by 2 rule is exact, so changing the rule to 3 by 3 will make no difference.

For nonrectangular elements no rule is exact, but 3 by 3 is more nearly exact than 2 by 2; it stiffens the elements and therefore reduces computed deflection.

6.8-6

Consider square el. $2a$ units / side.

$$[\underline{J}] = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, [\underline{J}]^{-1} = \frac{1}{a} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Eqs. 6.12-3:

$$u = 3\xi\eta^2 - \xi, \quad u_{,\xi} = 3\eta^2 - 1, \quad u_{,\eta} = 6\xi\eta$$

$$v = \eta - 3\xi^2\eta, \quad v_{,\xi} = -6\xi\eta, \quad v_{,\eta} = 1 - 3\xi^2$$

$$\epsilon_x = \frac{1}{a} u_{,\xi}, \quad \epsilon_y = \frac{1}{a} v_{,\eta}, \quad \gamma_{xy} = \frac{1}{a} (u_{,\eta} + v_{,\xi})$$

$$\gamma_{xy} = 0 \text{ for all } \xi, \eta$$

$$u_{,\xi} = v_{,\eta} = 0 \text{ for } \xi, \eta = \pm \frac{1}{\sqrt{3}}; \text{ i.e.}$$

$$\epsilon_x = \epsilon_y = 0 \text{ at Gauss points of } 2 \times 2 \text{ rule.}$$

6.8-7

(a) At $\xi = 0$, Eq. 6.1-7 gives $[B] = \frac{1}{J} \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$. Therefore

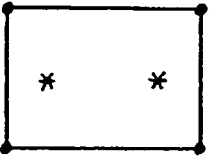
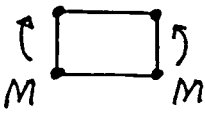
$\{\underline{d}\} = \begin{Bmatrix} 0 \\ u_2 \\ 0 \end{Bmatrix}$ is a spurious mode.

(b)  An inflection point (zero curvature) appears at midspan ($\xi = 0$).

At midspan, $x = \frac{L}{2}$, Eq. 3.3-13 yields $[B] = \begin{bmatrix} 0 & -\frac{1}{L} & 0 & \frac{1}{L} \end{bmatrix}$

$[B]\{\underline{d}\} = 0$ when $\{\underline{d}\} = \begin{Bmatrix} b \\ c \\ b \\ c \end{Bmatrix}$, so this $\{\underline{d}\}$ is a spurious mode.

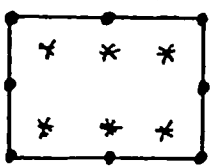
6.8-8

- (a)  8 d.o.f.
3 rigid body modes
no spurious modes:  M not resisted by ϵ_x
(but resisted by γ_{xy})
Rank 5

Disadvantages: not frame invariant

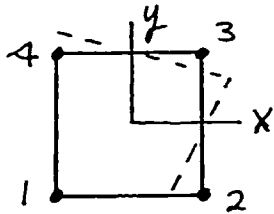
- (b) Like part (a), except that $\gamma_{xy} = 0$ on $\eta = 0$, so M is not resisted.
Rank 4

Disadvantages: one spurious mode, not frame invariant

- (c)  See problem 6.8-6: ϵ_y is now nonzero for
the mode of Fig. 6.8-3d, so no spurious mode
Rank 13

Disadvantages: not frame invariant

6.8-9



$$x = \xi$$

$$y = \eta$$

$$N = \frac{1}{4}(1 \pm x)(1 \pm y)$$

For the element shown,

$$\{d\} = [0, 0, -c, 0, c, -c, 0, c]^T$$

$$u = \sum N_i u_i = c(N_3 - N_2)$$

$$u = \frac{c}{4}(1+x)[(1+y) - (1-y)] = \frac{c}{4}(1+x)(2y)$$

$$v = \sum N_i v_i = c(N_4 - N_3)$$

$$v = \frac{c}{4}(1+y)[(1-x) - (1+x)] = \frac{c}{4}(1+y)(-2x)$$

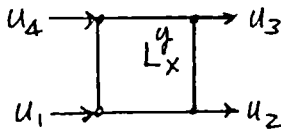
$$\epsilon_x = cy/2, \quad \epsilon_y = -cx/2$$

$$\gamma_{xy} = \frac{c}{2}[(1+x) - (1+y)] = \frac{c}{2}(x-y)$$

All zero
@ center
(x=y=0).

(b) Consider u first. $u = (\text{mode 7}) + \text{rigid-body rotation; i.e. at nodes}$

$$\begin{Bmatrix} 0 \\ -c \\ +c \\ 0 \end{Bmatrix} = \begin{Bmatrix} +b \\ -b \\ +b \\ -b \end{Bmatrix} + \theta \begin{Bmatrix} +1 \\ +1 \\ -1 \\ -1 \end{Bmatrix}$$

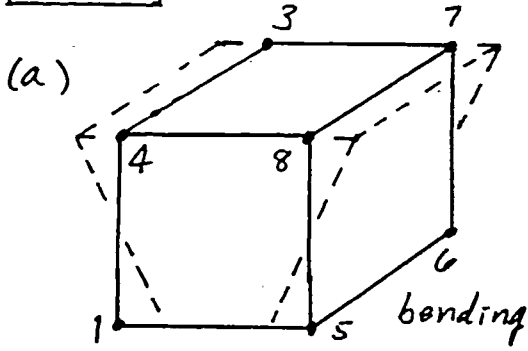


Works if $b = \frac{c}{2}$ and $\theta = -\frac{c}{2}$. Now check that these b & θ values also work for v_i .

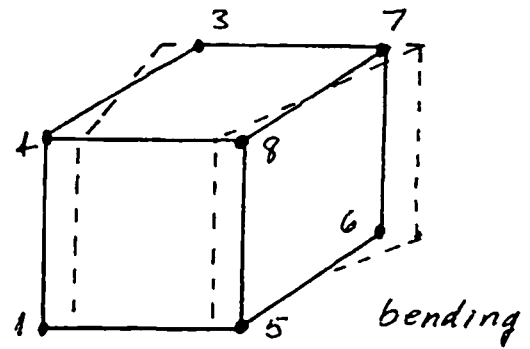
$$\begin{Bmatrix} 0 \\ 0 \\ -c \\ +c \end{Bmatrix} = \begin{Bmatrix} -c/2 \\ +c/2 \\ -c/2 \\ +c/2 \end{Bmatrix} - \frac{c}{2} \begin{Bmatrix} -1 \\ +1 \\ +1 \\ -1 \end{Bmatrix}$$

mode 8 ——— rotation ———
Checks.

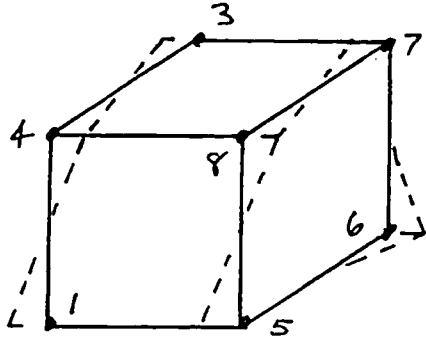
6,8-10



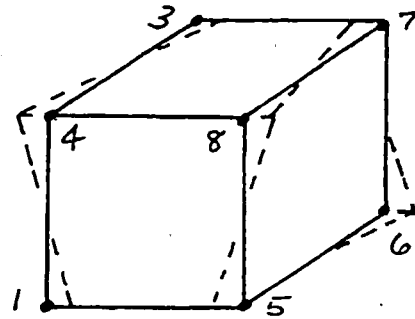
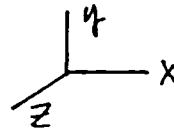
$$\{\underline{d}_x\}_1 = [1 \ 1 \ -1 \ -1 \ -1 \ -1 \ 1 \ 1]^T$$



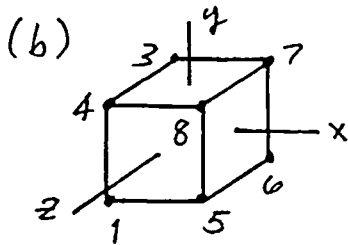
$$\{\underline{d}_x\}_2 = [1 \ -1 \ -1 \ 1 \ -1 \ 1 \ 1 \ -1]^T$$



$$\{\underline{d}_x\}_3 = [-1 \ 1 \ -1 \ 1 \ -1 \ 1 \ -1 \ 1]^T$$



$$\{\underline{d}_x\}_4 = [1 \ -1 \ 1 \ -1 \ -1 \ 1 \ -1 \ 1]^T$$



Translation: $\{\underline{d}_x\}_5 = [1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1]^T$

Const. ϵ_x : $\{\underline{d}_x\}_6 = [0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1]^T$

Const. γ_{xy} : $\{\underline{d}_x\}_7 = [-1 \ -1 \ 1 \ 1 \ -1 \ -1 \ 1 \ 1]^T$

Const. γ_{zx} : $\{\underline{d}_x\}_8 = [1 \ -1 \ -1 \ 1 \ 1 \ -1 \ -1 \ 1]^T$

(c) Want to show that $\{\underline{d}_x\}_i^T \{\underline{d}_x\}_j = 0$ for $i = 1, 2, 3, 4$ & $j = 5, 6, 7, 8$.
Straightforward calculation shows that this is so.

6.8-11

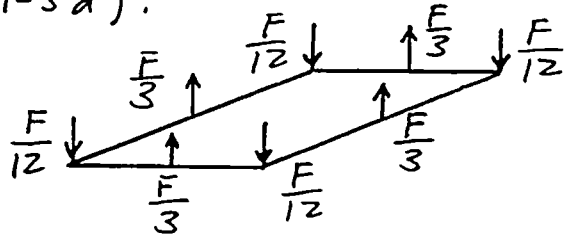
Weight factors: all must be equal, and to obtain volume = 8 for a 2 by 2 by 2 cube, $6W_i = 8$, so $W_i = 4/3$ ($i=1, 2, \dots, 6$).

Spurious modes: any deformation state for which in-plane strains are zero at the middle of each face. Mode 4 of Problem 6.8-10 is such a state (for example, $\epsilon_x = \epsilon_y = \gamma_{xy} = 0$ at the middle of face 1-5-4-8). There are two more such states, involving y- and z-direction nodal displacements respectively. Thus we expect three spurious modes, and a 24 by 24 matrix $[k]$ of rank 15. There are 6 rigid body modes, 6 constant strain modes, and three bending modes that store strain energy.

Yes, the spurious modes are communicable.

6.9-1

With F the total force applied, uniform pressure on an 8-node rectangular surface gives the following nodal loads (from Fig. 3.11-3 d):



For a nine-node rectangular surface,

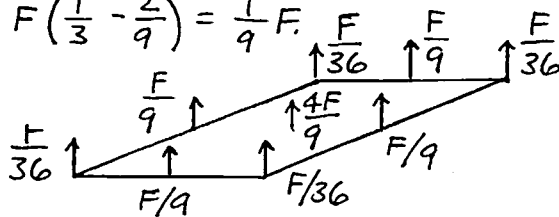
$$\int N_9 dA = \int_{-1}^1 \int_{-1}^1 (1-\xi^2)(1-\eta^2) d\xi d\eta$$

$$\int_{-1}^1 (1-\eta^2) d\eta = 2\left(\eta - \frac{\eta^3}{3}\right)_0^1 = \frac{4}{3}, \quad \int N_9 dA = \frac{16}{9}$$

If pressure p_0 uniform on 2×2 element, $F_9 = \frac{16}{9} p_0$. Let $F = \text{total force} = (2)(2)p_0$.

Thus $F_9 = \frac{4}{9} F$. In Table 6.4-1, contribution of N_9 to N_1 thru N_4 is $-\frac{1}{2}(-\frac{1}{2}N_9 - \frac{1}{2}N_9) - \frac{1}{4}N_9 = \frac{1}{4}N_9$. So, to corner loads in above Fig., add $\frac{1}{4}F_9 = \frac{1}{9}F$, for net result $F(-\frac{1}{12} + \frac{1}{9}) = \frac{1}{36}F$

In Table 6.4-1, contribution of N_9 to N_5 is $\frac{1}{4}N_9$. So, to midside loads in above Fig., add $-\frac{1}{2}F_9 = -\frac{2}{9}F$, for net result $F(\frac{1}{3} - \frac{2}{9}) = \frac{1}{9}F$.



6.9-2

$$(a) \{r_e\} = \int_0^L [N]^T q dx = q \int_{-1}^1 \begin{Bmatrix} \frac{1}{2}(-\xi + \xi^2) \\ 1 - \xi^2 \\ \frac{1}{2}(\xi + \xi^2) \end{Bmatrix} \frac{L}{2} d\xi = \frac{qL}{2} \begin{Bmatrix} \frac{1}{2}(-\frac{\xi^2}{2} + \frac{\xi^3}{3}) \\ \xi - \frac{\xi^3}{3} \\ \frac{1}{2}(\frac{\xi^2}{2} + \frac{\xi^3}{3}) \end{Bmatrix} \Big|_{-1}^1$$

$$\{r_e\} = \frac{qL}{2} \begin{Bmatrix} \frac{1}{2} \frac{2}{3} \\ \frac{4}{3} \\ \frac{1}{2} \frac{2}{3} \end{Bmatrix} = \frac{qL}{6} \begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}$$

$$(b) r_a = \int_0^L N_a q dx = q \int_{-1}^1 (1 + \cos \pi \xi) \frac{L}{2} d\xi = \frac{qL}{2} \left(\xi + \frac{\sin \pi \xi}{\pi} \right) \Big|_{-1}^1 = qL$$

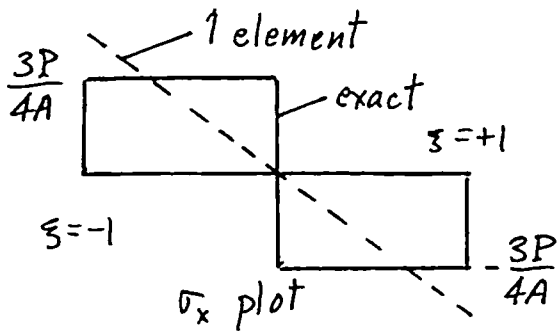
6.10-1

$$(a) \frac{16AE}{3L} u_2 = P, \quad u_2 = \frac{3PL}{16AE} \quad (\text{exact: } u_2 = \frac{(P/2)(L/2)}{AE} = \frac{PL}{4E})$$

$$\epsilon_x = \frac{dN_2}{dx} u_2 = \frac{1}{J} \frac{dN_2}{d\xi} u_2 = \frac{1}{L/2} (-2\xi) u_2 = -\frac{4\xi}{L} \frac{3PL}{16AE} = -\frac{3P}{4AE} \xi$$

$$\sigma_x = E\epsilon_x = -\frac{3P}{4A} \xi \quad (\text{exact: } \sigma_x = \frac{P}{2A} \text{ for } -1 < \xi < 0)$$

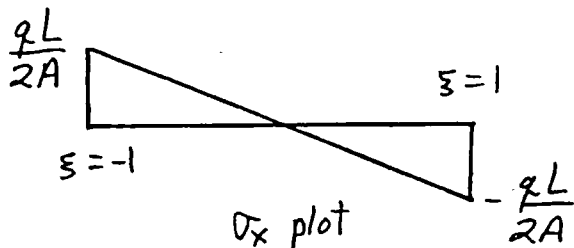
$$\sigma_x = -\frac{P}{2A} \text{ for } 0 < \xi < 1)$$



(b) From Problem 6.9-2a, load at node 2 is $\frac{2qL}{3}$

Part (a) repeats with P replaced by $\frac{2qL}{3}$. Thus

$$\sigma_x = -\frac{qL}{2A} \xi \quad (\text{this is exact})$$



6.10-2

$$(a) B_a = \frac{d^2}{dx^2} = \left[\frac{1}{(L/2)} \right]^2 \frac{d^2}{d\xi^2}$$

$$v = a_1 (1 + \cos \pi \xi)$$

$$k_a = EI \int_0^L B_a^2 dx = EI \int_{-1}^1 \left(\frac{-\pi^2 \cos \pi \xi}{(L/2)^2} \right)^2 \frac{L}{2} d\xi$$

$$k_a = \frac{8EI\pi^4}{L^3} \left(\frac{\xi}{2} + \frac{1}{4\pi} \sin 2\pi \xi \right)_{-1}^1 = \frac{8EI\pi^4}{L^3}$$

(b) As in Problem 6.9-2b,

$$r_a = \int_0^L N_a q dx = q \int_{-1}^1 (1 + \cos \pi \xi) \frac{L}{2} d\xi = \frac{qL}{2} \left(\xi + \frac{\sin \pi \xi}{\pi} \right)_{-1}^1 = qL$$

$$k_a a_1 = r_a, \quad a_1 = \frac{r_a}{k_a} = \frac{qL^4}{8\pi^4 EI}$$

$$\text{Exact center deflection: } \frac{qL^4}{384EI}$$

$$\text{Approx. center deflection: } v_0 = 2a_1 = \frac{qL^4}{4\pi^4 EI} \quad (1.45\% \text{ low})$$

$$(c) M = EI \frac{d^2 v}{dx^2} = EI \left(\frac{-\pi^2 \cos \pi \xi}{(L/2)^2} \right) a_1 = - \left(\frac{4\pi^2 EI}{L^2} \cos \pi \xi \right) a_1$$

$$|M_{\text{ends}}| = |M_{\text{center}}| = \frac{4\pi^2 EI}{L^2} a_1 = \frac{4\pi^2 EI}{L^2} \frac{qL^4}{8\pi^4 EI} = \frac{qL^2}{2\pi^2}$$

$$\text{Exact: } M_{\text{ends}} = \frac{qL^2}{12}, \quad M_{\text{center}} = -\frac{qL^2}{24} \quad (\text{for upward load})$$

Approx. M_{end} is 39.2% low

Approx. $|M_{\text{center}}|$ is 21.6% high

6.10-3

(a) As in Problem 6.10-2a, $k_a = \frac{8EI\pi^4}{L^3}$

(b) $v = N_a a_1$, where $N_a = 1 + \cos \pi \xi$. At center, $N_a = 2$

$$r_a = (N_a)_{\text{center}} P = 2P$$

$$k_a a_1 = r_a, \quad a_1 = \frac{r_a}{k_a} = \frac{PL^3}{4\pi^4 EI}$$

$$\text{Approx. center deflection: } v_0 = 2a_1 = \frac{PL^3}{2\pi^4 EI} = 0.005133 \frac{PL^3}{EI}$$

$$\text{Exact center deflection: } \frac{PL^3}{192EI} = 0.005208 \frac{PL^3}{EI} \quad \nearrow$$

1.45% low

(c) As in Problem 6.10-2c,

$$|M_{\text{ends}}| = |M_{\text{center}}| = \frac{4\pi^2 EI}{L^2} a_1 = \frac{PL}{\pi^2}$$

$$\text{Exact magnitudes are } \frac{PL}{8} \quad \nearrow$$

18.9% low

6.10-4

$$\delta_{xy} = u_{,y} + v_{,x} = u_{,y} \quad (\text{here } v=0)$$

For an el. $2a$ units on a side, $\xi = x/a$
and $\eta = y/a$, and $N = \frac{1}{4a^2} (a \pm x)(a \pm y)$

$$u_{,y} = \frac{1}{4a^2} \left[-(a-x)(-\bar{u}) - (a+x)(\bar{u}) \right. \\ \left. + (a+x)(-\bar{u}) + (a-x)(\bar{u}) \right]$$

where u = magnitude of corner d.o.f.

$$u_{,y} = \frac{1}{4a^2} (-4\bar{u}x) = -\frac{\bar{u}x}{a^2} = -\frac{\bar{u}}{a} \xi$$

6.10-5

$$(a,b) \quad N_1 = \frac{1}{4}(1-r)(1-s) \quad N_2 = \frac{1}{4}(1+r)(1-s) \\ N_3 = \frac{1}{4}(1+r)(1+s) \quad N_4 = \frac{1}{4}(1-r)(1+s)$$

	N_1	N_2	N_3	N_4
$r = -\sqrt{3}, s = -\sqrt{3}$	1.866	-0.5	0.134	-0.5
$r = +\sqrt{3}, s = -\sqrt{3}$	-0.5	1.866	-0.5	0.134
$r = +\sqrt{3}, s = +\sqrt{3}$	0.134	-0.5	1.866	-0.5
$r = -\sqrt{3}, s = +\sqrt{3}$	-0.5	0.134	-0.5	1.866

$$\begin{Bmatrix} \sigma_A \\ \sigma_B \\ \sigma_C \\ \sigma_D \end{Bmatrix} = \begin{bmatrix} \uparrow \\ \\ \\ \end{bmatrix}_{4 \times 4} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{Bmatrix}$$

This is result of evaluating $\sigma_p = \sum N_i \sigma_i$ at corners $P=A, P=B, P=C$, and $P=D$.

(c) Evaluate $\sigma_p = \sum N_i \sigma_i$ at midsides $P=E, P=F, P=G, P=H$:

	N_1	N_2	N_3	N_4
$r = 0, s = -\sqrt{3}$	a	a	b	b
$r = +\sqrt{3}, s = 0$	b	a	a	b
$r = 0, s = +\sqrt{3}$	b	b	a	a
$r = -\sqrt{3}, s = 0$	a	b	b	a

$$\begin{matrix} a = 0.683 \\ b = -0.183 \end{matrix} \quad \begin{Bmatrix} \sigma_E \\ \sigma_F \\ \sigma_G \\ \sigma_H \end{Bmatrix} = \begin{bmatrix} \uparrow \\ \\ \\ \end{bmatrix}_{4 \times 4} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{Bmatrix}$$

6.10-6

For extrapolation, Eq. 6.5-5 becomes $N = \frac{1}{8}(1 \pm r)(1 \pm s)(1 \pm t)$.

At a corner, with $r, s, t = \pm \sqrt{3}$,

$$\frac{1}{8}(1 + \sqrt{3})^3 = 2.549, \quad \frac{1}{8}(1 + \sqrt{3})^2(1 - \sqrt{3}) = -0.683$$

$$\frac{1}{8}(1 - \sqrt{3})^3 = -0.049, \quad \frac{1}{8}(1 + \sqrt{3})(1 - \sqrt{3})^2 = 0.183$$

At node 8 in Fig. 6.5-1a, with Gauss points given the number of the nearest corner node,

$$\sigma_{\text{node } 8} = 2.549\sigma_8 - 0.683(\sigma_4 + \sigma_5 + \sigma_7) \\ + 0.183(\sigma_1 + \sigma_3 + \sigma_6) - 0.049\sigma_2$$

Check: $\sigma_{\text{node } 8} = \bar{\sigma}$ if $\sigma_i = \bar{\sigma}$ for all i .

$$(b) N_i = \frac{1}{8}(1 - \sqrt{3})(1)^2 = -0.092 \quad i = 1, 2, 3, 4$$

$$N_i = \frac{1}{8}(1 + \sqrt{3})(1)^2 = 0.341 \quad i = 5, 6, 7, 8$$

$$\sigma_{\text{point}} = -0.092(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4) \\ + 0.341(\sigma_5 + \sigma_6 + \sigma_7 + \sigma_8)$$

Check: $\sigma_{\text{point}} = \bar{\sigma}$ if $\sigma_i = \bar{\sigma}$ for all i .

Because J isn't constant in a general element. Let's take an example; the 3-node bar of Fig. 6.1-1, with $u_1 = u_3 = 0$, $u_2 \neq 0$. Compute ϵ_x at node 1. By Eq. 6.1-7,

$$\epsilon_{x1} = \frac{1}{J} [N_{,1}] \begin{Bmatrix} 0 \\ 0 \\ u_3 \end{Bmatrix} = \frac{1}{J} \begin{bmatrix} -\frac{3}{2} & 2 & -\frac{1}{2} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_3 \end{Bmatrix} = \frac{-u_3}{2J}$$

At Gauss pts. of 2-pt. rule,

$$\left. \begin{aligned} \epsilon_{x*1} &= \frac{1}{J_1} \frac{1-2\sqrt{3}/3}{2} u_3 \\ \epsilon_{x*2} &= \frac{1}{J_2} \frac{1+2\sqrt{3}/3}{2} u_3 \end{aligned} \right\} \text{ for } u_1 = u_2 = 0$$

Extrapolate to node 1:

$$\epsilon_{x1} = \frac{1+\sqrt{3}}{2} \epsilon_{x*1} + \frac{1-\sqrt{3}}{2} \epsilon_{x*2}$$

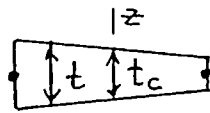
$$\epsilon_{x1} = \left(\frac{-0.105}{J_1} - \frac{0.395}{J_2} \right) u_3$$

Only for $J_1 = J_2 = J$ do the two ϵ_{x1} expressions yield the same result, viz.

$$\epsilon_{x1} = -0.5 \frac{u_3}{J} = -0.5 \frac{u_3}{L/2} = -\frac{u_3}{L}$$

6.10-8

For a bilinear element, t plays no role in Eq. 6.10-1; for the element



shown, ϵ_x is independent of x . An ad hoc adjustment that may yield better accuracy

is $\{\tilde{\epsilon}\} = \frac{t_c}{t} [B] \{d\}$, where t_c is thickness at the center (or apply t_c/t to adjust $\{\sigma\}$ from Eq. 6.10-1).

For a quadratic element, this adjustment should matter less, as side node



can displace relative to corners, thus automatically providing a strain variation.

6.10-9

$$(a) J = \left[\frac{1}{2}(-1+2\xi) \quad -2\xi \quad \frac{1}{2}(1+2\xi) \right] \begin{Bmatrix} 0 \\ 0.6L \\ L \end{Bmatrix} = -1.2L\xi + \frac{L}{2} + 2L\xi$$

$$J = \frac{L}{2}(1-0.4\xi)$$

$$\epsilon_x = \frac{1}{J} \left[- \quad - \quad \frac{1}{2}(1+2\xi) \right] \begin{Bmatrix} 0 \\ 0 \\ L/1000 \end{Bmatrix} = 0.001 \frac{1+2\xi}{1-0.4\xi}$$

Node 1, $\xi = -1$:

$$\epsilon_x = -0.000714$$

Node 2, $\xi = 0$:

$$\epsilon_x = 0.00100$$

Node 3, $\xi = 1$:

$$\epsilon_x = 0.00500$$

(b) 1st Gauss pt., $\xi = -1/\sqrt{3}$

$$\epsilon_x = -0.0001257$$

2nd Gauss pt., $\xi = 1/\sqrt{3}$

$$\epsilon_x = 0.002802$$

(c) Extrapolation from Gauss points: let $r = \sqrt{3}\xi$

$$\epsilon_x = \frac{1}{2} \begin{bmatrix} 1-r & 1+r \end{bmatrix} \begin{Bmatrix} -0.0001257 \\ 0.002802 \end{Bmatrix}$$

Node 1, $r = -\sqrt{3}$:

$$\epsilon_x = -0.00120$$

Node 2, $r = 0$:

$$\epsilon_x = 0.00134$$

Node 3, $r = \sqrt{3}$:

$$\epsilon_x = 0.00387$$

(d)

$$\epsilon_x = \frac{0.00500 + 0.00387}{3 - \frac{0.00500}{0.00387}} = 0.00519$$

G.10-10

$$(a) \quad [k] \{d\} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \end{Bmatrix} = \frac{AE}{L} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} u_2$$

$$\text{Eq. 6.10-7: } \sigma = \beta$$

$$\text{Eq. 6.10-9: } [Q] = \int_0^L \begin{Bmatrix} -1/L \\ 1/L \end{Bmatrix} (1) A dx = A \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$[Q]^T [Q] = 2A^2$$

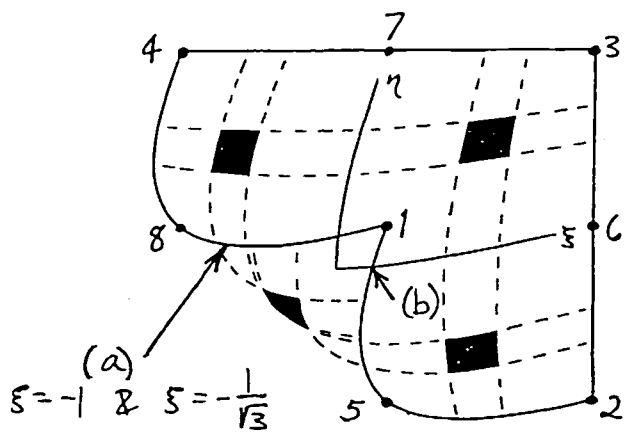
$$\text{Eq. 6.10-10: } (2A^2) \beta = A \begin{bmatrix} -1 & 1 \end{bmatrix} \frac{AE}{L} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} u_2$$

$$2A^2 \beta = \frac{A^2 E}{L} 2u_2$$

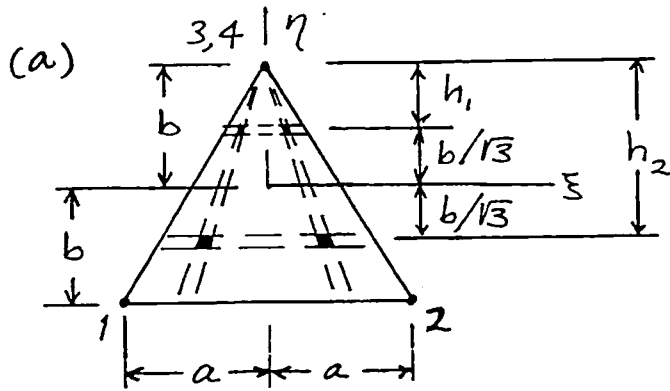
$$\beta = \frac{Eu_2}{L} \quad \text{so } \sigma = \beta = E \frac{u_2}{L} \quad \checkmark$$

(b) In our development, we require equilibrium, which for the bar requires $\frac{d\sigma_x}{dx} = 0$. Not satisfied by $\sigma_x = \beta_1 + \beta_2 x$.

6.11-1



6.11-2



(b) Eq. 6.2-6:

$$[\underline{J}] = \frac{1}{4} \begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi) \end{bmatrix} \begin{bmatrix} -a & -b \\ a & -b \\ 0 & b \\ 0 & b \end{bmatrix}$$

$$[\underline{J}] = \frac{1}{4} \begin{bmatrix} 2(1-\eta)a & 0 \\ -2\xi a & 4b \end{bmatrix}, \quad J = \det[\underline{J}] = 2ab(1-\eta)$$

Smallest J in element is $J=0$ at nodes 3,4
(as is obvious by inspection).

J at Gauss points:

At $\eta = -\frac{1}{\sqrt{3}}$, $J = 3.1547ab$ (call it $J_{\max}$)

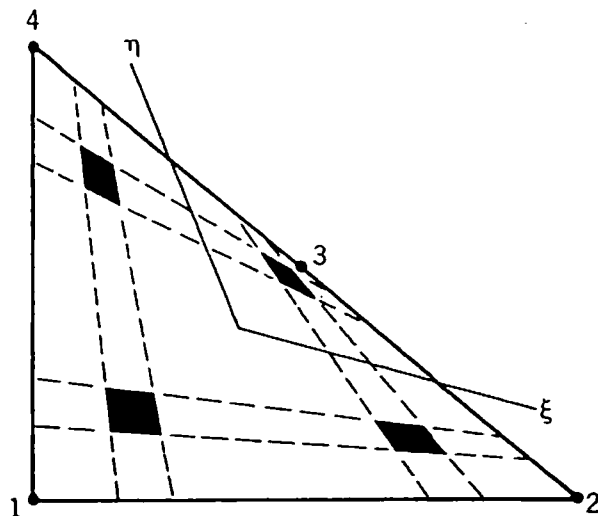
At $\eta = \frac{1}{\sqrt{3}}$, $J = 0.8453ab$ (call it $J_{\min}$)

$\therefore \frac{J_{\max}}{J_{\min}} = \frac{3.1547}{0.8453} = 3.732$

Or, more directly, $\frac{J_{\max}}{J_{\min}} = \frac{h_2}{h_1} = \frac{b + b/\sqrt{3}}{b - b/\sqrt{3}} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1} = 3.732$

6.11-3

(a)



$$(b) \quad [J] = \frac{1}{4} \begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi) \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & -1 \\ 0 & 0 \\ -1 & 1 \end{bmatrix}$$

$$[J] = \frac{1}{4} \begin{bmatrix} 3-\eta & -1-\eta \\ -1-\xi & 3-\xi \end{bmatrix}, \quad J = \det[J] = 2-\xi-\eta$$

$$\text{Node 1: } \xi = -1, \eta = -1; \quad J = 4$$

$$\text{Node 2: } \xi = 1, \eta = -1; \quad J = 2$$

$$\text{Node 3: } \xi = 1, \eta = 1; \quad J = 0$$

$$\text{Node 4: } \xi = -1, \eta = 1; \quad J = 2$$

6.12-1

(a) 4-node element:

$$\begin{aligned}\sum N_i &= \frac{1}{4}(1+\xi)[(1-\eta)+(1+\eta)] + \frac{1}{4}(1-\xi)[(1-\eta)+(1+\eta)] \\ &= \frac{1}{2}(1+\xi) + \frac{1}{2}(1-\xi) = 1\end{aligned}$$

8-node element:

From the above we know that bilinear portions of N_1 through N_4 in the N_i of Eqs. 6.4-1 sum to unity. Also, by inspection, the higher-order terms cancel in the sum of all eight N_i :

$$(b) \sum N_{i,\xi} = \frac{1}{4}[-(1-\eta) + (1-\eta) + (1+\eta) - (1+\eta)] = 0$$

$$\sum N_{i,\eta} = \frac{1}{4}[-(1-\xi) - (1+\xi) + (1+\xi) + (1-\xi)] = 0$$

6.12-2

$$J = \begin{bmatrix} \frac{1}{2}(-1+2\xi) & -2\xi & \frac{1}{2}(1+2\xi) \end{bmatrix} \begin{Bmatrix} 0 \\ x_2 \\ L \end{Bmatrix} = \frac{L}{2}(1+2\xi) - 2\xi x_2$$

For linear $u = u(x)$,

$$u = \begin{bmatrix} \frac{1-\xi}{2} & \frac{1+\xi}{2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_3 \end{Bmatrix}, \quad \frac{du}{d\xi} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_3 \end{Bmatrix} = \frac{u_3 - u_1}{2}$$

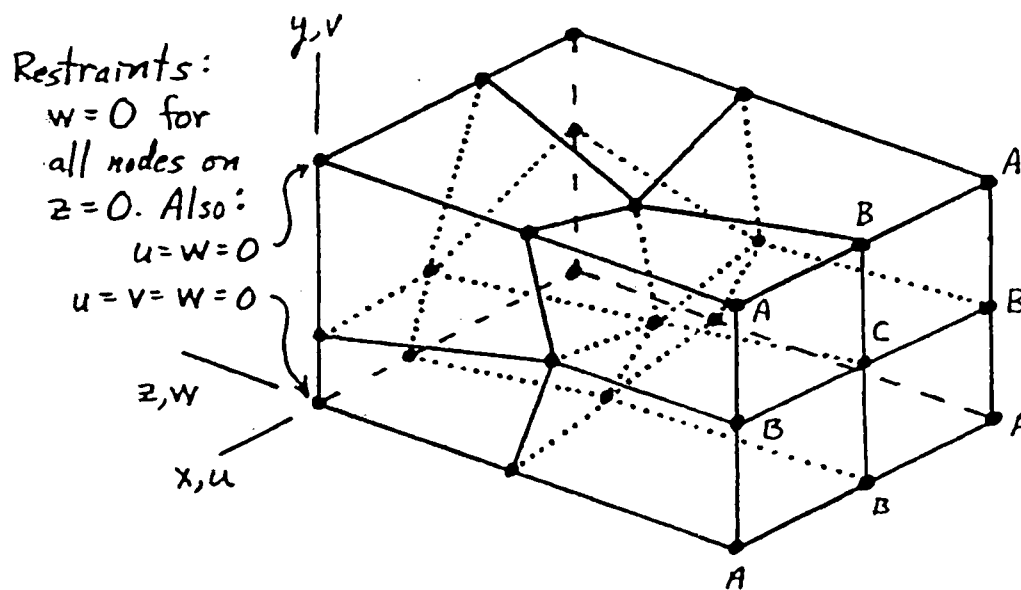
$$\epsilon_x = \frac{du}{d\xi} \frac{d\xi}{dx} = \frac{du/d\xi}{dx/d\xi} = \frac{1}{J} \frac{du}{d\xi} = \frac{u_3 - u_1}{L(1+2\xi) - 4\xi x_2}$$

We obtain the expected $\epsilon_x = \frac{u_3 - u_1}{L}$ only if $x_2 = \frac{L}{2}$ (the midpoint)

6.13-1

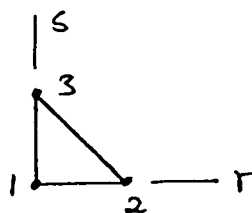
Zero. In "standard" patch test, only boundary nodes loaded, & $\{D\} = [K]^{-1}\{R\}$ gives $\{D\}$ consistent with a const. strain state. Therefore, using this $\{D\}$, $[K]\{D\}$ gives an $\{R\}$ with loads on boundary nodes only.

6.13-2



Eight arbitrarily shaped hexahedra fill a rectangular box, with one completely internal node. Rectangular faces on right end suggest z -direction loads of 1 @ A, 2 @ B, 4 @ C.

7.1-1



$$\phi = a_1 + a_2 r + a_3 s$$

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = [A] \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

Solve for the a_i :

$$a_1 = \phi_1$$

$$a_2 = \phi_2 - \phi_1$$

$$a_3 = \phi_3 - \phi_1$$

$$\text{hence } \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}$$

$$\phi = [1 \quad r \quad s] [A]^{-1} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = [1-r-s \quad r \quad s] \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}$$

7.1-2

$$x = \begin{bmatrix} N_1 & N_2 & \dots & N_6 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ (x_1+x_2)/2 \\ (x_2+x_3)/2 \\ (x_3+x_1)/2 \end{Bmatrix}$$

$$x = \begin{bmatrix} (N_1 + \frac{N_4}{2} + \frac{N_6}{2}) & (N_2 + \frac{N_4}{2} + \frac{N_5}{2}) & (N_3 + \frac{N_5}{2} + \frac{N_6}{2}) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

$$\begin{aligned} N_1 + \frac{N_4}{2} + \frac{N_6}{2} &= (1-r-s)(1-2r-2s) + 2r(1-r-s) + 2s(1-r-s) \\ &= (1-r-s)(1-2r-2s+2r+2s) = 1-r-s \quad \checkmark \end{aligned}$$

$$\begin{aligned} N_2 + \frac{N_4}{2} + \frac{N_5}{2} &= r(2r-1) + 2r(1-r-s) + 2rs \\ &= 2r^2 - r + 2r - 2r^2 - 2rs + 2rs = r \quad \checkmark \end{aligned}$$

$$\begin{aligned} N_3 + \frac{N_5}{2} + \frac{N_6}{2} &= s(2s-1) + 2rs + 2s(1-r-s) \\ &= 2s^2 - s + 2rs + 2s - 2rs - 2s^2 = s \quad \checkmark \end{aligned}$$

7.1-3

Obtain i th shape function by taking products of functions which, if equated to zero, are equations of lines that do not pass through the i th node. Multiply each such product by a constant c_i such that it becomes unity at node i .

$$c_1 N_1 = \left(\frac{1}{3} - r - s\right) \left(\frac{2}{3} - r - s\right) (1 - r - s); \quad c_1 N_1 = \frac{2}{9} \text{ for } r=s=0, \text{ so } c_1 = \frac{2}{9}$$

$$N_1 = (1 - 3r - 3s) \left(1 - \frac{3}{2}r - \frac{3}{2}s\right) (1 - r - s)$$

$$c_2 N_2 = r \left(\frac{1}{3} - r\right) \left(\frac{2}{3} - r\right); \quad c_2 N_2 = \frac{2}{9} \text{ for } r=1, s=0, \text{ so } c_2 = \frac{2}{9}$$

$$N_2 = r(1 - 3r) \left(1 - \frac{3r}{2}\right)$$

$$c_4 N_4 = r(1 - r - s) \left(\frac{2}{3} - r - s\right); \quad c_4 N_4 = \frac{2}{27} \text{ for } r=\frac{1}{3}, s=0, \text{ so } c_4 = \frac{2}{27}$$

$$N_4 = 9r(1 - r - s) \left(1 - \frac{3r}{2} - \frac{3s}{2}\right)$$

$$c_5 N_5 = r(1 - r - s) \left(\frac{1}{3} - r\right); \quad c_5 N_5 = -\frac{2}{27} \text{ for } r=\frac{2}{3}, s=0, \text{ so } c_5 = -\frac{2}{27}$$

$$N_5 = 9r(1 - r - s) \left(\frac{3r}{2} - \frac{1}{2}\right)$$

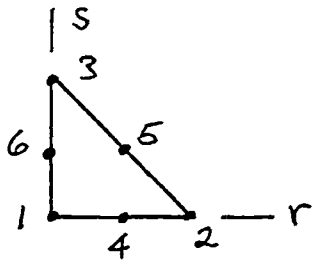
$$c_6 N_6 = rs \left(\frac{1}{3} - r\right); \quad c_6 N_6 = \frac{2}{3} \cdot \frac{1}{3} \left(\frac{1}{3} - \frac{2}{3}\right) = -\frac{2}{27} \text{ for } r=\frac{2}{3}, s=\frac{1}{3}; \quad c_6 = -\frac{2}{27}$$

$$N_6 = \frac{9}{2} rs(3r - 1)$$

$$c_{10} N_{10} = rs(1 - r - s); \quad c_{10} N_{10} = \frac{1}{27} \text{ for } r=s=\frac{1}{3}, \text{ so } c_{10} = \frac{1}{27}$$

$$N_{10} = 27rs(1 - r - s)$$

7.1-4



The Table:

Include only if node i is present in the element

	<u>$i=4$</u>	<u>$i=5$</u>	<u>$i=6$</u>
$N_1 = 1-r-s$	$-\frac{1}{2}N_4$		$-\frac{1}{2}N_6$
$N_2 = r$	$-\frac{1}{2}N_4$	$-\frac{1}{2}N_5$	
$N_3 = s$		$-\frac{1}{2}N_5$	$-\frac{1}{2}N_6$

$$N_4 = 4r(1-r-s)$$

$$N_5 = 4rs$$

$$N_6 = 4s(1-r-s)$$

} [from Eqs. 7.1-2]

If nodes 4, 5, 6 are all present, then

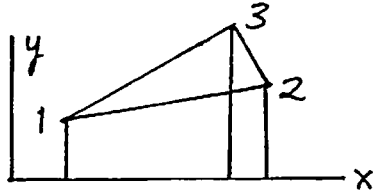
$$N_1 = 1-r-s - 2r(1-r-s) - 2s(1-r-s) = (1-r-s)(1-2r-2s)$$

$$N_2 = r - 2r(1-r-s) - 2rs = -r + 2r^2 = r(2r-1)$$

$$s - 2s(1-r-s) = -s + 2s^2 = s(2s-1)$$

Checks N_1, N_2, N_3 in Eqs. 7.1-2.

7.2-1



$A = \text{area of triangle}$
 $= (2 \text{ "tall" trapezoids}) \text{ minus}$
 $(1 \text{ "short" trapezoid})$

$$A = \frac{y_1 + y_3}{2}(x_3 - x_1) + \frac{y_2 + y_3}{2}(x_2 - x_3) - \frac{y_1 + y_2}{2}(x_2 - x_1)$$

$$2A = x_3 y_1 + x_3 y_3 - x_1 y_1 - x_1 y_3 + x_2 y_2 + x_2 y_3 - x_3 y_2 - x_3 y_3 \\ + x_1 y_1 + x_1 y_2 - x_2 y_1 - x_2 y_2$$

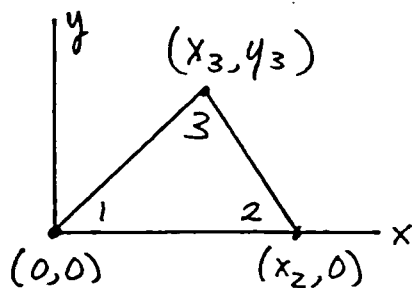
$$2A = x_2(y_3 - y_1) - x_1(y_3 - y_2) - x_3(y_2 - y_1)$$

$$2A = x_2(y_3 - y_1) - x_1(y_3 - y_1) - x_1 y_1 + x_1(y_2 - y_1) + x_1 y_1 - x_3(y_2 - y_1)$$

$$2A = (x_2 - x_1)(y_3 - y_1) + (x_1 - x_3)(y_2 - y_1)$$

$$2A = x_{21} y_{31} - x_{31} y_{21} \quad \text{which is } \det \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ from Eq. 7.2-3}$$

7.2-2



Eq. 7.2-4:

$$J = x_{21}y_{31} - x_{31}y_{21}$$

$$x_{21} = x_2 - x_1 = x_2$$

$$y_{31} = y_3 - y_1 = y_3$$

$$x_{31} = x_3 - x_1 = x_3$$

$$y_{21} = y_2 - y_1 = 0$$

Hence $J = 2A = x_2 y_3$

Eq. 7.2-6:

$$[B]_{\sim} = \frac{1}{2A} \begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix} = \frac{1}{x_2 y_3} \begin{bmatrix} -y_3 & y_3 & 0 \\ x_3 - x_2 & -x_3 & x_2 \end{bmatrix}$$

$$[B]_{\sim} = \begin{bmatrix} -\frac{1}{x_2} & \frac{1}{x_2} & 0 \\ \frac{x_3 - x_2}{x_2 y_3} & -\frac{x_3}{x_2 y_3} & \frac{1}{y_3} \end{bmatrix}$$

Checks Eq. 3.4-6

7.2-3

$$\phi = N_1\phi_1 + N_2\phi_2 + N_3\phi_3 + N_4\phi_4 = (N_1 + N_4)\phi_1 + N_2\phi_2 + N_3\phi_3$$

(for a quadrilateral), $N_1 + N_4 = \frac{1}{2}(1-\xi)$, so

$$\phi = \frac{1}{2}(1-\xi)\phi_1 + \frac{1}{4}(1+\xi)(1-\eta)\phi_2 + \frac{1}{4}(1+\xi)(1+\eta)\phi_3$$

Derivatives of shape functions: define

$$[\underline{D}_N] = \begin{bmatrix} \phi_{1,\xi} & \phi_{2,\xi} & \phi_{3,\xi} \\ \phi_{1,\eta} & \phi_{2,\eta} & \phi_{3,\eta} \end{bmatrix} = \begin{bmatrix} -1/2 & (1-\eta)/4 & (1+\eta)/4 \\ 0 & -(1+\xi)/4 & (1+\xi)/4 \end{bmatrix}$$

$$[\underline{J}] = [\underline{D}_N] \begin{bmatrix} 0 & 0 \\ a & 0 \\ 0 & b \end{bmatrix} = \frac{1}{4} \begin{bmatrix} (1-\eta)a & (1+\eta)b \\ -(1+\xi)a & (1+\xi)b \end{bmatrix}$$

$$[\underline{\Gamma}] = [\underline{J}]^{-1} = \frac{2}{(1+\xi)ab} \begin{bmatrix} (1+\xi)b & -(1+\eta)b \\ (1+\xi)a & (1-\eta)a \end{bmatrix}$$

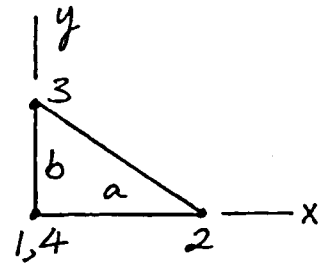
$$\begin{Bmatrix} \phi_x \\ \phi_y \end{Bmatrix} = [\underline{\Gamma}][\underline{D}_N] \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = [\underline{B}] \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} \quad \text{where}$$

$$[\underline{B}] = \frac{1}{(1+\xi)ab} \begin{bmatrix} -(1+\xi)b & \frac{b}{2}(1+\xi)(1-\eta) + \frac{b}{2}(1+\xi)(1+\eta) & \frac{b}{2}(1+\xi)(1+\eta) - \frac{b}{2}(1+\xi)(1-\eta) \\ -(1+\xi)a & \frac{a}{2}(1+\xi)(1-\eta) - \frac{a}{2}(1+\xi)(1+\eta) & \frac{a}{2}(1+\xi)(1+\eta) + \frac{a}{2}(1+\xi)(1-\eta) \end{bmatrix}$$

$$[\underline{B}] = \frac{1}{(1+\xi)ab} \begin{bmatrix} -(1+\xi)b & (1+\xi)b & 0 \\ -(1+\xi)a & 0 & (1+\xi)a \end{bmatrix} = \begin{bmatrix} -1/a & 1/a & 0 \\ -1/b & 0 & 1/b \end{bmatrix}$$

Eq. 7.2-6:

$$[\underline{B}] = \frac{1}{ab} \begin{bmatrix} -b & b & 0 \\ -a & 0 & a \end{bmatrix} = \begin{bmatrix} -1/a & 1/a & 0 \\ -1/b & 0 & 1/b \end{bmatrix}$$



7.3-1

$$\sum_1^6 N_i = 2(\xi_1^2 + \xi_2^2 + \xi_3^2) - (\xi_1 + \xi_2 + \xi_3) + 4(\xi_1 \xi_2 + \xi_2 \xi_3 + \xi_3 \xi_1)$$

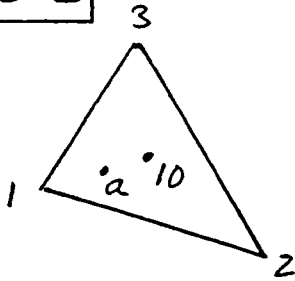
Call the above Eq. (a). Now

$$(\xi_1 + \xi_2 + \xi_3)^2 = (\xi_1^2 + \xi_2^2 + \xi_3^2) + 2(\xi_1 \xi_2 + \xi_2 \xi_3 + \xi_3 \xi_1)$$

If we multiply this equation by 2 and subtract $(\xi_1 + \xi_2 + \xi_3)$, we get the right hand side of Eq. (a). But since $\xi_1 + \xi_2 + \xi_3 = 1$, this means that

$$\sum_1^6 N_i = 2(1) - 1 \quad \text{so} \quad \sum_1^6 N_i = 1$$

7.3-2



Point a has coordinates $\xi_1 = \frac{2}{3}$, $\xi_2 = \xi_3 = \frac{1}{6}$

$$\phi_a = 27 \frac{2}{3} \frac{1}{6} \frac{1}{6} \phi_{10}, \text{ so } \phi_{10} = 2 \phi_a$$

$$\text{and } \phi = 54 \xi_1 \xi_2 \xi_3 \phi_a$$

$$V = \int_A 54 \xi_1 \xi_2 \xi_3 dA = 54 (2A) \frac{\phi_a}{(2+1+1+1)!} = \frac{108}{5!} A \phi_a = 0.9 A \phi_a$$

7.3-3

7.9 | $X = x_1 \xi_1 + x_2 \xi_2 + x_3 \xi_3$

$$\int x^2 dA = \int (x_1^2 \xi_1^2 + x_2^2 \xi_2^2 + x_3^2 \xi_3^2 + 2x_1 x_2 \xi_1 \xi_2 + 2x_2 x_3 \xi_2 \xi_3 + 2x_3 x_1 \xi_3 \xi_1) dA$$

Eq. 5.2-8 yields

$$\int \xi_i^2 dA = \frac{A}{6} \quad \& \quad \int \xi_i \xi_j dA = \frac{A}{12} \quad \text{Hence}$$

$$\int x^2 dA = \frac{A}{6} (x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_2 x_3 + x_3 x_1) \quad (a)$$

Add $x_1 x_2 - x_1 x_2$ to right side & factor

$$\int x^2 dA = \frac{A}{6} [x_1 (x_1 + x_2 + x_3) + x_2 (x_1 + x_2 + x_3) - x_1 x_2 + x_3^2]$$

But $x_1 + x_2 + x_3 = 0$ because of centroidal coordinates, so

$$\int x^2 dA = \frac{A}{6} (x_3^2 - x_1 x_2) = \frac{A}{6} [x_3 (-x_1 - x_2) - x_1 x_2] = \frac{A}{6} (-x_1 x_2 - x_2 x_3 - x_3 x_1) \quad (b)$$

From (a) and (b),

$$\int x^2 dA = \frac{A}{6} (x_1^2 + x_2^2 + x_3^2) - \int x^2 dA \quad \text{hence} \quad \int x^2 dA = \frac{A}{12} (x_1^2 + x_2^2 + x_3^2)$$

7.3-4

Evaluate the given ϕ equation at nodes.

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 0 & 0 \\ 0 & 1/4 & 1/4 & 0 & 1/4 & 0 \\ 1/4 & 0 & 1/4 & 0 & 0 & 1/4 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

Hence $a_1 = \phi_1$, $a_2 = \phi_2$, $a_3 = \phi_3$.

Subs. these into the last 3 eqs.

$$\begin{aligned} a_4 &= 4\phi_4 - \phi_1 - \phi_2 \\ a_5 &= 4\phi_5 - \phi_2 - \phi_3 \\ a_6 &= 4\phi_6 - \phi_3 - \phi_1 \end{aligned} \left\{ \begin{aligned} \phi &= \phi_1 \xi_1^2 + \phi_2 \xi_2^2 + \phi_3 \xi_3^2 \\ &+ (4\phi_4 - \phi_1 - \phi_2) \xi_1 \xi_2 \\ &+ (4\phi_5 - \phi_2 - \phi_3) \xi_2 \xi_3 \\ &+ (4\phi_6 - \phi_3 - \phi_1) \xi_3 \xi_1 \end{aligned} \right.$$

$$\begin{aligned} \phi &= (\xi_1^2 - \xi_1 \xi_2 - \xi_3 \xi_1) \phi_1 + (\xi_2^2 - \xi_1 \xi_2 - \xi_2 \xi_3) \phi_2 \\ &+ (\xi_3^2 - \xi_2 \xi_3 - \xi_3 \xi_1) \phi_3 + 4(\xi_1 \xi_2 \phi_4 + \xi_2 \xi_3 \phi_5 + \xi_3 \xi_1 \phi_6) \end{aligned}$$

Now use relation $\xi_1 + \xi_2 + \xi_3 = 1$.

$$\begin{aligned} \xi_1^2 - \xi_1 \xi_2 - \xi_3 \xi_1 &= \xi_1 (\xi_1 - \xi_2 - \xi_3) = \xi_1 [\xi_1 - (1 - \xi_1)] \\ &= \xi_1 (2\xi_1 - 1) \end{aligned}$$

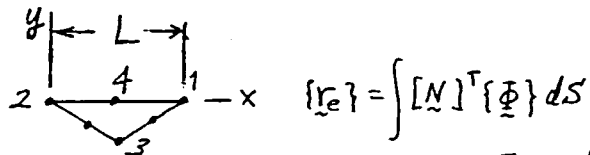
Similarly,

$$\xi_2^2 - \xi_1 \xi_2 - \xi_2 \xi_3 = \xi_2 (2\xi_2 - 1)$$

$$\xi_3^2 - \xi_2 \xi_3 - \xi_3 \xi_1 = \xi_3 (2\xi_3 - 1). \text{ Finally,}$$

$$\begin{aligned} \phi &= \xi_1 (2\xi_1 - 1) \phi_1 + \xi_2 (2\xi_2 - 1) \phi_2 + \xi_3 (2\xi_3 - 1) \phi_3 \\ &+ 4\xi_1 \xi_2 \phi_4 + 4\xi_2 \xi_3 \phi_5 + 4\xi_3 \xi_1 \phi_6 \end{aligned}$$

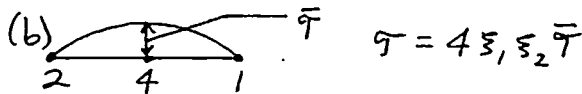
7.3-5



(a) $\begin{Bmatrix} r_{y1} \\ r_{y4} \\ r_{y2} \end{Bmatrix} = \begin{Bmatrix} \xi_1(2\xi_1-1) \\ 4\xi_1\xi_2 \\ \xi_2(2\xi_2-1) \end{Bmatrix} p t dL$ Use Eq. 7.3-5 to integrate

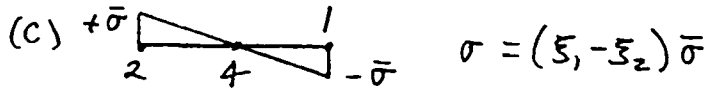
$$\int \xi_i^2 dL = \frac{L}{3}, \int \xi_i dL = \frac{L}{2}, \int \xi_i \xi_j dL = \frac{L}{6}$$

$$\begin{Bmatrix} r_{y1} \\ r_{y4} \\ r_{y2} \end{Bmatrix} = p t L \begin{Bmatrix} \frac{2}{3} - \frac{1}{2} \\ \frac{4}{6} \\ \frac{2}{3} - \frac{1}{2} \end{Bmatrix} = \frac{p t L}{6} \begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}$$



$$\int \xi_1^2 \xi_2 dL = \frac{L}{12}, \int \xi_1^3 \xi_2 dL = \frac{L}{20}, \int \xi_1^2 \xi_2^2 dL = \frac{L}{30}$$

$$\begin{Bmatrix} r_{x1} \\ r_{x4} \\ r_{x2} \end{Bmatrix} = \begin{Bmatrix} \xi_1(2\xi_1-1) \\ 4\xi_1\xi_2 \\ \xi_2(2\xi_2-1) \end{Bmatrix} 4\xi_1\xi_2\bar{\gamma} t dL = \frac{\bar{\gamma} t L}{15} \begin{Bmatrix} 1 \\ 8 \\ 1 \end{Bmatrix}$$



$$\begin{Bmatrix} r_{y1} \\ r_{y4} \\ r_{y2} \end{Bmatrix} = \begin{Bmatrix} \xi_1(2\xi_1-1) \\ 4\xi_1\xi_2 \\ \xi_2(2\xi_2-1) \end{Bmatrix} (\xi_1 - \xi_2)\bar{\sigma} t dL$$

$$\int \xi_1^3 dL = \frac{L}{4}, \int \xi_1^2 \xi_2^2 dL = \frac{L}{6}$$

$$\begin{Bmatrix} r_{y1} \\ r_{y4} \\ r_{y2} \end{Bmatrix} = \bar{\sigma} t \begin{Bmatrix} 2\frac{L}{4} - \frac{L}{3} - 2\frac{L}{12} + \frac{L}{6} \\ 4(\frac{L}{12} - \frac{L}{12}) \\ 2\frac{L}{12} - \frac{L}{6} - 2\frac{L}{4} + \frac{L}{3} \end{Bmatrix} = \bar{\sigma} t L \begin{Bmatrix} \frac{1}{6} \\ 0 \\ -\frac{1}{6} \end{Bmatrix}$$

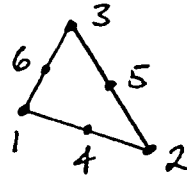
Yields $M = 2 \frac{L}{2} \left(\frac{\bar{\sigma} t L}{6} \right) = \frac{\bar{\sigma} t L^2}{6}$

Check by flexure formula:

$$M = \frac{\sigma I}{c} = \frac{\bar{\sigma} \frac{1}{12} t L^3}{L/2} = \frac{\bar{\sigma} t L^2}{6} \checkmark$$

7.3-6

For one face, we deal with the triangle



For constant p , on face of area A ,

$$\{r_e\} = \int [N]^T p dA = p \int [N]^T dA$$

where the N_i come from Eq. 7.3-4.

Use Eq. 7.3-7 for integration:

$$\int \xi_i dA = \frac{A}{3} \quad \int \xi_i^2 dA = \frac{A}{6} \quad \int \xi_i \xi_j dA = \frac{A}{12}$$

$$\{r_e\} = p \frac{A}{12} \begin{Bmatrix} 2(2) - 4 \\ 2(2) - 4 \\ 2(2) - 4 \\ 4 \\ 4 \\ 4 \end{Bmatrix} = \frac{pA}{3} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$$

7.4-1 $\phi = a_1 r + a_2 r^2 + a_3 rs + a_4 r^3 + a_5 rs^2$, straight sides, $A = \frac{1}{2}$

(a) In area coordinates, $\phi = a_1 \xi_2 + a_2 \xi_2^2 + a_3 \xi_2 \xi_3 + a_4 \xi_2^3 + a_5 \xi_2 \xi_3^2$
Using Eq. 7.3-7,

$$\begin{aligned} \int \phi dA &= a_1 \frac{1}{6} + a_2 \frac{2}{4!} + a_3 \frac{1}{4!} + a_4 \frac{3!}{5!} + a_5 \frac{2}{5!} \\ &= \frac{a_1}{6} + \frac{a_2}{12} + \frac{a_3}{24} + \frac{a_4}{20} + \frac{a_5}{60} \\ &= 0.1667a_1 + 0.08333a_2 + 0.04167a_3 + 0.05000a_4 + 0.01667a_5 \end{aligned}$$

(b) Formula 1, Table 7.4-1, $|\underline{J}| = 2A = 1$

$$\int \phi dA \approx \frac{1}{2} \left(\frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^2} + \frac{a_4}{3^3} + \frac{a_5}{3^3} \right) = \frac{a_1}{6} + \underbrace{\frac{a_2}{18} + \frac{a_3}{18} + \frac{a_4}{54} + \frac{a_5}{54}}_{\text{approx.}}$$

(c) Formula 2, Table 7.4-1, $|\underline{J}| = 2A = 1$

$$\begin{aligned} \int \phi dA &\approx \frac{1}{2} \frac{1}{3} \left[a_1 \frac{2}{3} + a_2 \left(\frac{2}{3} \right)^2 + a_3 \left(\frac{2}{3} \right) \left(\frac{1}{6} \right) + a_4 \left(\frac{2}{3} \right)^3 + a_5 \left(\frac{2}{3} \right) \left(\frac{1}{6} \right)^2 \right] \\ &\quad + \frac{1}{2} \frac{1}{3} \left[a_1 \frac{1}{6} + a_2 \left(\frac{1}{6} \right)^2 + a_3 \left(\frac{1}{6} \right)^2 + a_4 \left(\frac{1}{6} \right)^3 + a_5 \left(\frac{1}{6} \right)^3 \right] \\ &\quad + \frac{1}{2} \frac{1}{3} \left[a_1 \frac{1}{6} + a_2 \left(\frac{1}{6} \right)^2 + a_3 \left(\frac{1}{6} \right) \left(\frac{2}{3} \right) + a_4 \left(\frac{1}{6} \right)^3 + a_5 \left(\frac{1}{6} \right) \left(\frac{2}{3} \right)^2 \right] \\ &= \frac{1}{6} \left[a_1 \left(\frac{2}{3} + \frac{1}{6} + \frac{1}{6} \right) + a_2 \left(\frac{4}{9} + \frac{1}{36} + \frac{1}{36} \right) + a_3 \left(\frac{2}{18} + \frac{1}{36} + \frac{2}{18} \right) \right. \\ &\quad \left. + a_4 \left(\frac{8}{27} + \frac{1}{216} + \frac{1}{216} \right) + a_5 \left(\frac{2}{108} + \frac{1}{216} + \frac{4}{54} \right) \right] \\ &= 0.1667a_1 + 0.08333a_2 + 0.04167a_3 + \underbrace{0.05093a_4 + 0.01620a_5}_{\text{approx.}} \end{aligned}$$

(d) Formula 3, Table 7.4-1, $|\underline{J}| = 2A = 1$ approx.

$$\begin{aligned} \int \phi dA &\approx \frac{1}{5} \frac{1}{3} \left[a_1 \frac{1}{2} + a_2 \left(\frac{1}{2} \right)^2 + a_3 (0) + a_4 \left(\frac{1}{2} \right)^3 + a_5 (0) \right] \\ &\quad + \frac{1}{2} \frac{1}{3} \left[a_1 (0) + a_2 (0) + a_3 (0) + a_4 (0) + a_5 (0) \right] \\ &\quad + \frac{1}{2} \frac{1}{3} \left[a_1 \frac{1}{2} + a_2 \left(\frac{1}{2} \right)^2 + a_3 \left(\frac{1}{2} \right)^2 + a_4 \left(\frac{1}{2} \right)^3 + a_5 \left(\frac{1}{2} \right)^3 \right] \\ &= \frac{1}{6} \left[a_1 + a_2 \left(\frac{1}{2} \right) + a_3 \left(\frac{1}{4} \right) + a_4 \left(\frac{1}{4} \right) + a_5 \left(\frac{1}{8} \right) \right] \\ &= 0.1667a_1 + 0.08333a_2 + 0.04167a_3 + \underbrace{0.04167a_4 + 0.02083a_5}_{\text{approx.}} \end{aligned}$$

(continues)

approx.

7.4-1 (concluded)

(e) Formula 4, Table 7.4-1, $|J| = 2A = 1$

$$\begin{aligned}
 \int \phi dA &\approx \frac{1}{2} \left(-\frac{27}{48} \right) \left[a_1 \left(\frac{1}{3} \right) + a_2 \left(\frac{1}{3} \right)^2 + a_3 \left(\frac{1}{3} \right)^2 + a_4 \left(\frac{1}{3} \right)^3 + a_5 \left(\frac{1}{3} \right)^3 \right] \\
 &\quad + \frac{1}{2} \left(\frac{25}{48} \right) \left[a_1 \left(\frac{3}{5} \right) + a_2 \left(\frac{3}{5} \right)^2 + a_3 \left(\frac{3}{5} \right) \left(\frac{1}{5} \right) + a_4 \left(\frac{3}{5} \right)^3 + a_5 \left(\frac{3}{5} \right) \left(\frac{1}{5} \right)^2 \right] \\
 &\quad + \frac{1}{2} \left(\frac{25}{48} \right) \left[a_1 \left(\frac{1}{5} \right) + a_2 \left(\frac{1}{5} \right)^2 + a_3 \left(\frac{1}{5} \right)^2 + a_4 \left(\frac{1}{5} \right)^3 + a_5 \left(\frac{1}{5} \right)^3 \right] \\
 &\quad + \frac{1}{2} \left(\frac{25}{48} \right) \left[a_1 \left(\frac{1}{5} \right) + a_2 \left(\frac{1}{5} \right)^2 + a_3 \left(\frac{1}{5} \right) \left(\frac{3}{5} \right) + a_4 \left(\frac{1}{5} \right)^3 + a_5 \left(\frac{1}{5} \right) \left(\frac{3}{5} \right)^2 \right] \\
 &= \frac{1}{96} \left[a_1 (-9 + 15 + 5 + 5) + a_2 (-3 + 9 + 1 + 1) + a_3 (-3 + 3 + 1 + 3) \right. \\
 &\quad \left. + a_4 \left(-1 + \frac{27}{5} + \frac{1}{5} + \frac{1}{5} \right) + a_5 \left(-1 + \frac{3}{5} + \frac{1}{5} + \frac{9}{5} \right) \right] \\
 &= 0.1667a_1 + 0.08333a_2 + 0.04167a_3 + 0.0500a_4 + 0.01667a_5 \\
 &\quad \checkmark \quad \quad \checkmark \quad \quad \checkmark \quad \quad \checkmark \quad \quad \checkmark
 \end{aligned}$$

7.4-2

$$1\text{-pt. } I \approx (1) \frac{1}{1 + \frac{1}{3} \frac{1}{3}} = \frac{1}{\frac{10}{9}} = 0.9000$$

$$1^{\text{st}} 3\text{-pt. } I \approx \frac{1}{3} \left[\frac{1}{1 + \frac{2}{3} \frac{1}{6}} + \frac{1}{1 + \frac{1}{6} \frac{1}{6}} + \frac{1}{1 + \frac{1}{6} \frac{2}{3}} \right] = 0.924324$$

$$2^{\text{nd}} 3\text{-pt. } I \approx \frac{1}{3} \left[\frac{1}{1+0} + \frac{1}{1+0} + \frac{1}{1+\frac{1}{4}} \right] = 0.933333$$

$$4\text{-pt. } I \approx -0.5625 \frac{1}{1 + \frac{1}{9}} + 0.5208333 \left[\frac{1}{1+.12} + \frac{1}{1+.04} + \frac{1}{1+.12} \right] = 0.924611$$

7.4-3

$$\text{Eq. 7.2-2: } [\underline{J}] = \begin{bmatrix} x_{,r} & y_{,r} \\ x_{,s} & y_{,s} \end{bmatrix} \quad \begin{aligned} x &= \sum N_i x_i \\ y &= \sum N_i y_i \end{aligned}$$

Contents of $[\underline{J}]$ are linear in r and s , so $|\underline{J}|$ is quadratic in r and s . By Eq. 7.4-1

$$\text{Volume} = \int t dA = \sum_{i=1}^n \frac{1}{2} |\underline{J}|_i t_i W_i$$

where t is quadratic in r and s . Hence integrand contains 4th powers of r and s . Need degree of precision = 4.

7.4-4 $\phi = a_1 r + a_2 r^2 + a_3 rs + a_4 r^3 + a_5 rs^2 + a_6 rst$ Flat faces, $V = \frac{1}{6}$

(a) In volume coords., $\phi = a_1 \xi_2 + a_2 \xi_2^2 + a_3 \xi_2 \xi_3 + a_4 \xi_2^3 + a_5 \xi_2 \xi_3^2 + a_6 \xi_2 \xi_3 \xi_4$
Using Eq. 7.3-9,

$$\begin{aligned} \int \phi dV &= a_1 \frac{1}{4!} + a_2 \frac{2}{5!} + a_3 \frac{1}{5!} + a_4 \frac{3!}{6!} + a_5 \frac{2}{6!} + a_6 \frac{1}{6!} \\ &= 0.04167a_1 + 0.01667a_2 + 0.008333a_3 \\ &\quad + 0.008333a_4 + 0.002778a_5 + 0.001389a_6 \end{aligned}$$

(b) Formula 1, Table 7.4-2, $|\underline{j}| = 6V = 1$

$$\begin{aligned} \int \phi dV &\approx \frac{1}{6} \left[\frac{a_1}{4} + \frac{a_2}{16} + \frac{a_3}{16} + \frac{a_4}{4^3} + \frac{a_5}{4^3} + \frac{a_6}{4^3} \right] \\ &= 0.04167a_1 + 0.01042(a_2 + a_3) + 0.002604(a_4 + a_5 + a_6) \end{aligned}$$

(c) Formula 2, Table 7.4-2, $|\underline{j}| = 6V = 1$

$$a = \frac{5+3\sqrt{5}}{20} = 0.58541, \quad b = \frac{5-\sqrt{5}}{20} = 0.13820$$

$$\begin{aligned} \int \phi dV &\approx \frac{1}{4} \frac{1}{6} \left[a_1 a + a_2 a^2 + a_3 b^2 + a_4 b^3 + a_5 b^3 + a_6 a b^2 \right] \\ &\quad + \frac{1}{4} \frac{1}{6} \left[a_1 b + a_2 b^2 + a_3 b^2 + a_4 b^3 + a_5 b^3 + a_6 b^3 \right] \\ &\quad + \frac{1}{4} \frac{1}{6} \left[a_1 b + a_2 b^2 + a_3 a b + a_4 b^3 + a_5 a^2 b + a_6 a b^2 \right] \\ &\quad + \frac{1}{4} \frac{1}{6} \left[a_1 b + a_2 b^2 + a_3 a b + a_4 a^3 + a_5 a b^2 + a_6 a b^2 \right] \\ &= \frac{1}{24} \left[a_1 (a + 3b) + a_2 (a^2 + 3b^2) + a_3 (2ab + 2b^2) \right. \\ &\quad \left. + a_4 (a^3 + 3b^3) + a_5 (a^2 b + a b^2 + 2b^3) + a_6 (3ab^2 + b^3) \right] \\ &= 0.04167a_1 + 0.01667a_2 + 0.008333a_3 \\ &\quad + 0.008689a_4 + 0.002659a_5 + 0.001508a_6 \end{aligned}$$

(continues)

7.4-4 (concluded)

(d) Formula 3, Table 7.4-2, $|J| = 6V = 1$

$$\begin{aligned}
 \int \phi dV &\approx \left(-\frac{4}{5}\right) \frac{1}{6} \left[\frac{a_1}{4} + \frac{a_2}{4^2} + \frac{a_3}{4^2} + \frac{a_4}{4^3} + \frac{a_5}{4^3} + \frac{a_6}{4^3} \right] \\
 &\quad + \frac{9}{20} \frac{1}{6} \left[a_1 \frac{1}{2} + a_2 \frac{1}{4} + a_3 \left(\frac{1}{2}\right) \left(\frac{1}{6}\right) + a_4 \left(\frac{1}{2}\right)^3 + a_5 \left(\frac{1}{2}\right) \left(\frac{1}{6}\right)^2 + a_6 \left(\frac{1}{2}\right) \left(\frac{1}{6}\right)^2 \right] \\
 &\quad + \frac{9}{20} \frac{1}{6} \left[a_1 \frac{1}{6} + a_2 \left(\frac{1}{6}\right)^2 + a_3 \left(\frac{1}{6}\right)^2 + a_4 \left(\frac{1}{6}\right)^3 + a_5 \left(\frac{1}{6}\right)^3 + a_6 \left(\frac{1}{6}\right)^3 \right] \\
 &\quad + \frac{9}{20} \frac{1}{6} \left[a_1 \frac{1}{6} + a_2 \left(\frac{1}{6}\right)^2 + a_3 \left(\frac{1}{6}\right)^2 + a_4 \left(\frac{1}{6}\right)^3 + a_5 \left(\frac{1}{6}\right)^3 + a_6 \left(\frac{1}{6}\right)^2 \left(\frac{1}{2}\right) \right] \\
 &\quad + \frac{9}{20} \frac{1}{6} \left[a_1 \frac{1}{6} + a_2 \left(\frac{1}{6}\right)^2 + a_3 \left(\frac{1}{6}\right) \left(\frac{1}{2}\right) + a_4 \left(\frac{1}{6}\right)^3 + a_5 \left(\frac{1}{6}\right) \left(\frac{1}{2}\right)^2 + a_6 \left(\frac{1}{6}\right)^2 \left(\frac{1}{2}\right) \right] \\
 &= a_1 \left(-\frac{1}{30} + \frac{9}{240} + 3 \frac{9}{20(36)} \right) + a_2 \left(-\frac{1}{120} + \frac{9}{480} + 3 \frac{9}{20(36)6} \right) \\
 &\quad + a_3 \left(-\frac{1}{120} + 2 \frac{9}{20(2)6^2} + 2 \frac{9}{20(6^3)} \right) \\
 &\quad + a_4 \left(-\frac{1}{5(6)16} + \frac{9}{20(6)8} + 3 \frac{9}{20(6)6^3} \right) \\
 &\quad + a_5 \left(-\frac{1}{5(6)16} + \frac{9}{20(6)2(36)} + 2 \frac{9}{20(6^4)} + \frac{9}{20(36)4} \right) \\
 &\quad + a_6 \left(-\frac{1}{5(6)16} + 3 \frac{9}{20(6)2(36)} + \frac{9}{20(6)6^3} \right) \\
 &= 0.04167a_1 + 0.01667a_2 + 0.008333a_3 \\
 &\quad + 0.008333a_4 + 0.002778a_5 + 0.001389a_6
 \end{aligned}$$

8.1-1

$$L^2 = \underline{V} \cdot \underline{V} = [u' \ v' \ w'] \begin{Bmatrix} u' \\ v' \\ w' \end{Bmatrix} = \{\underline{d}'\}^T \{\underline{d}'\}$$

$$L^2 = ([\underline{\Lambda}]\{\underline{d}\})^T ([\underline{\Lambda}]\{\underline{d}\}). \text{ Also, } L^2 = \{\underline{d}\}^T \{\underline{d}\}.$$

$$\{\underline{d}\}^T [\underline{\Lambda}]^T [\underline{\Lambda}] \{\underline{d}\} = \{\underline{d}\}^T \{\underline{d}\}, \text{ or}$$

$$\{\underline{d}\}^T ([\underline{\Lambda}]^T [\underline{\Lambda}] - [\underline{I}]) \{\underline{d}\} = 0.$$

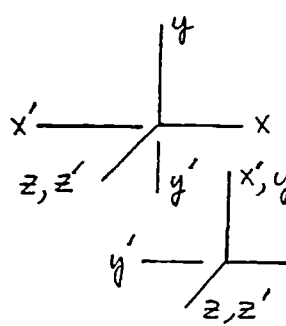
Must be true for any $\{\underline{d}\}$, so

$$[\underline{\Lambda}]^T [\underline{\Lambda}]^T = [\underline{I}]. \text{ From Eq. 8.1-1, this}$$

$$\text{means } \sum l_i^2 = \sum m_i^2 = \sum n_i^2 = 1$$

$$\text{and } \sum l_i m_i = \sum m_i n_i = \sum n_i l_i = 0$$

8.1-2



Top diagram: A 3D coordinate system with axes x, y, z is rotated around the y -axis to a new system x', y', z' . The y -axis is vertical, x is horizontal to the right, and z is diagonal down-left. The new axes x' and z' are rotated towards the y -axis.

Bottom diagram: A 3D coordinate system with axes x, y, z is rotated around the x -axis to a new system x', y', z' . The x -axis is horizontal to the right, y is vertical, and z is diagonal down-left. The new axes y' and z' are rotated towards the x -axis.

$$[\tilde{\Lambda}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\tilde{\Lambda}] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8.2-1

$$[\underline{E}'] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

We can ignore the common multiplier $E/(1-\nu^2)$ in the following. The product $[\underline{T}_\epsilon]^T [\underline{E}'] [\underline{T}_\epsilon]$ is found to be symmetric, with the following terms in its upper triangle. Let $c = \cos \theta$, $s = \sin \theta$.

$$(1,1) = c^4 + \nu c^2 s^2 + \nu c^2 s^2 + s^4 + 2c^2 s^2 - 2\nu c^2 s^2 \\ = (c^2 + s^2)^2 = 1$$

$$(1,2) = c^2 s^2 + \nu c^4 + \nu s^4 + c^2 s^2 - 2c^2 s^2 + 2\nu c^2 s^2 \\ = \nu (c^2 + s^2)^2 = \nu$$

$$(1,3) = c^3 s(1-\nu) + cs^3(\nu-1) - (1-\nu)(c^3 s - cs^3) \\ = 0$$

$$(2,2) = s^4 + \nu c^2 s^2 + \nu c^2 s^2 + c^4 + 2c^2 s^2(1-\nu) \\ = (c^2 + s^2)^2 = 1$$

$$(2,3) = cs^3(1-\nu) + c^3 s(\nu-1) + (1-\nu)(c^3 s - cs^3) \\ = 0$$

$$(3,3) = c^2 s^2(1-\nu) + c^2 s^2(1-\nu) + \frac{1-\nu}{2}(c^2 - s^2)^2 \\ = \frac{1-\nu}{2}(c^2 + s^2)^2 = \frac{1-\nu}{2}$$

So, after restoring the multiplier $E/(1-\nu^2)$, we obtain $[\underline{E}] = [\underline{E}']$.

8.2-2

$$[\underline{E}'] = \begin{bmatrix} E_a & 0 & 0 \\ 0 & E_b & 0 \\ 0 & 0 & G \end{bmatrix} \quad \begin{array}{c} y' \\ y \\ x' \\ x \end{array} \quad \begin{array}{l} c = \cos \beta \\ s = \sin \beta \end{array}$$

$[\underline{T}_\epsilon]^T ([\underline{E}']) [\underline{T}_\epsilon] =$

$$\begin{bmatrix} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ cs & -cs & c^2 - s^2 \end{bmatrix} \begin{bmatrix} c^2 E_a & s^2 E_a & cs E_a \\ s^2 E_b & c^2 E_b & -cs E_b \\ -2cs G & 2cs G & G(c^2 - s^2) \end{bmatrix} = [\underline{E}]$$

$$E_{11} = c^4 E_a + s^4 E_b + 4c^2 s^2 G$$

$$E_{12} = c^2 s^2 (E_a + E_b) - 4c^2 s^2 G = E_{21}$$

$$E_{13} = c^3 s E_a - cs^3 E_b - 2cs(c^2 - s^2)G = E_{31}$$

$$E_{22} = s^4 E_a + c^4 E_b + 4c^2 s^2 G$$

$$E_{23} = cs^3 E_a - c^3 s E_b + 2cs(c^2 - s^2)G = E_{32}$$

$$E_{33} = c^2 s^2 (E_a + E_b) + G(c^2 - s^2)^2$$

$$\text{For } \beta = 0, [\underline{E}] = [\underline{E}'] \quad \checkmark$$

$$\text{For } \beta = \pi/2, E_{11} = E_b, E_{22} = E_a, E_{33} = G, \\ \text{remainder of } [\underline{E}] \text{ null. } \checkmark$$

8.2-3

$$[\underline{T}_\epsilon]^T [\underline{T}_\epsilon] = \begin{bmatrix} c^4 + s^4 + 4c^2 s^2 & \dots \\ \vdots & \end{bmatrix}$$

$$c^4 + s^4 + 4c^2 s^2 = (c^2 + s^2)^2 + 2c^2 s^2 = 1 + c^2 s^2$$

Unity only if $c^2 s^2 = 0$; not so for all β

Hence $[\underline{T}_\epsilon]^T [\underline{T}_\epsilon] \neq [\underline{I}]$; $[\underline{T}_\epsilon]$ not orthogonal.

8.3-1

For d.o.f. u_1 & u_2 , $[k'] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$u_r = u_2 - u_1, \quad u_2 = u_1 + u_r$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{[I]} \begin{Bmatrix} u_1 \\ u_r \end{Bmatrix} \quad [I][k'] = \frac{AE}{L} \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$$

$$[I]^T ([k'] [I]) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \frac{AE}{L} \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

8.3-2

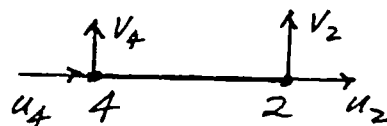
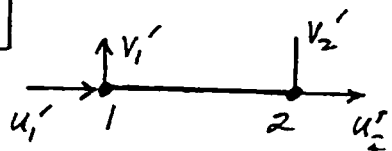
$$u = [N'] \{d'\} = \begin{bmatrix} \frac{-\zeta + \zeta^2}{2} & 1 - \zeta^2 & \frac{\zeta + \zeta^2}{2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad \text{or} \quad \{d'\} = [T] \{d\}$$

$$u = [N'] [T] \{d\} \quad \text{or} \quad u = [N] \{d\} \quad \text{where}$$

$$[N] = [N'] [T] = \begin{bmatrix} \frac{-\zeta + \zeta^2 + 1 - \zeta^2}{2} & 1 - \zeta^2 & \frac{1 - \zeta^2 + \zeta + \zeta^2}{2} \\ \frac{1 - \zeta}{2} & 1 - \zeta^2 & \frac{1 + \zeta}{2} \end{bmatrix}$$

8.3-3



$$\begin{Bmatrix} u_1' \\ v_1' \\ u_2' \\ v_2' \end{Bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}}_{[T]} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ \vdots \\ u_4 \\ v_4 \end{Bmatrix}$$

8.4-1

Set $u_1 = v_1 = u_2 = v_2 = 0$, but temporarily retain u_4 . $[K']\{Q'\} = \{R'\}$ is

$$k \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} R_{x3} \\ R_{y3} \\ R_{x4} \\ -P \end{Bmatrix} \quad \begin{array}{l} \text{Now set} \\ u_4 = 0: \\ \text{discard} \\ \text{row \& col. 3} \end{array}$$

$$\frac{k}{2} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} R_{x3} \\ R_{y3} \\ -P \end{Bmatrix}, \quad [T] = \begin{bmatrix} 0.8 & 0.6 & 0 \\ 0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T]^T ([K'] [T]) = \begin{bmatrix} 0.8 & -0.6 & 0 \\ 0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{k}{2} \begin{bmatrix} 1.8 & 2.6 & -1 \\ 0.2 & 1.4 & -1 \\ -0.2 & -1.4 & 3 \end{bmatrix}$$

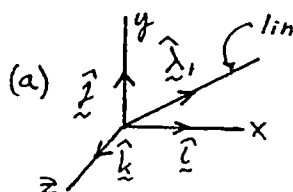
$$\frac{k}{2} \begin{bmatrix} 1.32 & 1.24 & -0.2 \\ 1.24 & 2.68 & -1.4 \\ -0.2 & -1.4 & 3.0 \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ R_3 \\ -P \end{Bmatrix} \quad \begin{array}{l} \text{Set } v_3 = 0, \\ \text{solve for} \\ u_3 \text{ \& } v_4. \end{array}$$

$$1.32 u_3 - 0.2 v_4 = 0, \quad u_3 = -0.102 P/k$$

$$-0.2 u_3 + 3.0 v_4 = -2P/k, \quad v_4 = -0.673 P/k$$

$$F_{1-3} = |k(u_3 \cos \beta)| = k(0.102 \frac{P}{k})(0.8) = 0.0816P$$

(tension)

(a)  $\hat{\lambda}_1 = l_1 \hat{i} + l_2 \hat{j} + l_3 \hat{k}$
 Set up orthogonal vectors $\underline{V}_2$ & $\underline{V}_3$. Arbitrarily define $\underline{V}_2 = \hat{\lambda}_1 \times \hat{i}$
 (make another choice if $\hat{\lambda}_1 = \hat{i}$).

$$\underline{V}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ l_1 & l_2 & l_3 \\ 1 & 0 & 0 \end{vmatrix} = l_3 \hat{j} - l_2 \hat{k}$$

Length of $\underline{V}_2$ is L_2 ,
 $L_2 = (l_2^2 + l_3^2)^{1/2}$

$$\underline{V}_3 = \hat{\lambda}_1 \times \underline{V}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ l_1 & l_2 & l_3 \\ 0 & l_3 & -l_2 \end{vmatrix} = (-l_2^2 - l_3^2) \hat{i} + l_1 l_2 \hat{j} + l_1 l_3 \hat{k}$$

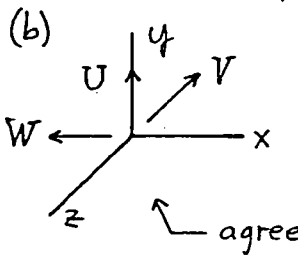
$$L_3 = [(-l_2^2 - l_3^2)^2 + (l_1 l_2)^2 + (l_1 l_3)^2]^{1/2}$$

Orthog. unit vectors are $\hat{\lambda}_1, \underline{V}_2/L_2, \underline{V}_3/L_3$.

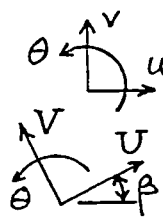
At affected node n , like $[\underline{\Lambda}]$ in Eq. 8.1-1,

$$[\underline{T}_n] = \begin{bmatrix} l_1 & 0 & (-l_2^2 - l_3^2)/L_3 \\ l_2 & l_3/L_2 & l_1 l_2/L_3 \\ l_3 & -l_2/L_2 & l_1 l_3/L_3 \end{bmatrix}$$

Put this in matrix $[\underline{T}]$ similar to Eq. 8.4-3.

(b)  $l_2 = 1, l_1 = l_3 = 0$
 $[\underline{T}_n] = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$
 agree

8.4-3

$$(a) \begin{bmatrix} \frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} u \\ v \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ 0 \end{Bmatrix}$$


$$\begin{Bmatrix} u \\ v \\ \theta \end{Bmatrix} = \underbrace{\begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[T]} \begin{Bmatrix} U \\ V \\ \theta \end{Bmatrix} \quad \begin{aligned} c &= \cos \beta \\ s &= \sin \beta \end{aligned}$$

$$\underbrace{\begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[T]^T} \underbrace{\begin{bmatrix} c \frac{AE}{L} & -s \frac{AE}{L} & 0 \\ s \frac{12EI}{L^3} & c \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ -s \frac{6EI}{L^2} & -c \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}}_{[K'] [T]} \begin{Bmatrix} U \\ V \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ 0 \end{Bmatrix}$$

"Omit middle" in product (i.e. set $V=0$)

$$\begin{bmatrix} c^2 \frac{AE}{L} + s^2 \frac{12EI}{L^3} & -s \frac{6EI}{L^2} \\ -s \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} U \\ \theta \end{Bmatrix} = \begin{Bmatrix} -sP \\ 0 \end{Bmatrix}$$

$$(b) U = -sP \left[c^2 \frac{AE}{L} + s^2 \frac{12EI}{L^3} \right]^{-1}$$

8.5-1

Substitute $\xi, \eta = \pm \frac{1}{2}$ in bilinear shape functions, Eqs. 6.2-3, e.g.

$$u_A = \frac{1}{4} \left(\frac{3}{2} \right) \left(\frac{3}{2} \right) u_1 + \frac{1}{4} \left(\frac{1}{2} \right) \left(\frac{3}{2} \right) u_2 + \frac{1}{4} \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) u_3 + \frac{1}{4} \left(\frac{3}{2} \right) \left(\frac{1}{2} \right) u_4$$

$$u_A = \frac{1}{16} (9u_1 + 3u_2 + u_3 + 3u_4)$$

$$\begin{Bmatrix} u_A \\ u_B \\ u_C \\ v_A \\ v_B \\ v_C \end{Bmatrix} = \begin{bmatrix} \tilde{I}_1 & 0 \\ 0 & \tilde{I}_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix}, \text{ where } [\tilde{I}_1] \text{ is } \frac{1}{16} \begin{bmatrix} 9 & 3 & 1 & 3 \\ 3 & 9 & 3 & 1 \\ 1 & 3 & 9 & 3 \end{bmatrix}$$

8.5-2

$$\{r\} = [T]^T \{r'\} = \frac{1}{L} \begin{bmatrix} a & 0 & c \\ 0 & a & s \\ b & 0 & -c \\ 0 & b & -s \end{bmatrix} \begin{Bmatrix} F_x \\ F_y \\ m_s \end{Bmatrix} \quad \begin{aligned} c &= \cos \beta \\ s &= \sin \beta \end{aligned}$$

(result)

$$\begin{aligned} & \frac{b}{L} F_y - s \frac{m_s}{L} \\ & \frac{b}{L} F_x - c \frac{m_s}{L} \\ & \frac{a}{L} F_x + c \frac{m_s}{L} \\ & \frac{a}{L} F_y + s \frac{m_s}{L} \end{aligned}$$



$$L = a + b$$

Now want to show that these two are stat. equiv.

$$F_x = \frac{b}{L} F_x - c \frac{m_s}{L} + \frac{a}{L} F_x + c \frac{m_s}{L} = \frac{a+b}{L} F_x = F_x$$

$$F_y = \frac{b}{L} F_y - s \frac{m_s}{L} + \frac{a}{L} F_y + s \frac{m_s}{L} = \frac{a+b}{L} F_y = F_y$$

$$\begin{aligned} m_s &= bc \left(\frac{a}{L} F_x + c \frac{m_s}{L} \right) - ac \left(\frac{b}{L} F_x - c \frac{m_s}{L} \right) \\ &\quad + bs \left(\frac{a}{L} F_y + s \frac{m_s}{L} \right) - as \left(\frac{b}{L} F_y - s \frac{m_s}{L} \right) \\ &= \left[(a+b)c^2 + (a+b)s^2 \right] \frac{m_s}{L} = (a+b) \frac{m_s}{L} = m_s \end{aligned}$$

8.5-3

$$u_5 = \frac{c}{L_2} u_1 + \frac{d}{L_2} u_4 \quad u_6 = \frac{a}{L_1} u_2 + \frac{b}{L_1} u_3$$

$$v_5 = \frac{c}{L_2} v_1 + \frac{d}{L_2} v_4 \quad v_6 = \frac{a}{L_1} v_2 + \frac{b}{L_1} v_3$$

$$\begin{bmatrix} u_5 & v_5 & u_6 & v_6 \end{bmatrix}^T = \begin{bmatrix} T \\ \sim \end{bmatrix} \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{bmatrix}^T$$

where $\begin{bmatrix} T \\ \sim \end{bmatrix}$ is the 4 by 8 matrix

$$\begin{bmatrix} c/L_2 & 0 & 0 & 0 & 0 & 0 & d/L_2 & 0 \\ 0 & c/L_2 & 0 & 0 & 0 & 0 & 0 & d/L_2 \\ 0 & 0 & a/L_1 & 0 & b/L_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a/L_1 & 0 & b/L_1 & 0 & 0 \end{bmatrix}$$

(a) $\{\underline{d}_1\} = [\underline{T}_1]\{\underline{d}_2\}$, where $[\underline{T}_1]$ is

$$\begin{bmatrix} 1 & 0 & H/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & H/2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -H/2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & -H/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This transformation sets $v_1 = v_4$ and $v_2 = v_3$.

(b) $\{\underline{d}_2\} = [\underline{T}_2]\{\underline{d}_1\}$, where $[\underline{T}_2]$ is

$$\begin{bmatrix} 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1/H & 0 & 0 & 0 & 0 & 0 & -1/H & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/H & 0 & -1/H & 0 & 0 & 0 \end{bmatrix}$$

This transformation averages the d.o.f. at each end of el. 1 to get the translational d.o.f. of el. 2, and says $\theta_1 = \frac{v_1 - v_4}{H}$, $\theta_2 = \frac{v_2 - v_3}{H}$.

(c) $\{\underline{d}_1\} = [\underline{T}_1]\{\underline{d}_2\} = [\underline{T}_1][\underline{T}_2]\{\underline{d}_1\}$

$$\{\underline{d}_2\} = [\underline{T}_2]\{\underline{d}_1\} = [\underline{T}_2][\underline{T}_1]\{\underline{d}_2\}$$

Hence we expect that $[\underline{T}_1][\underline{T}_2]$ and $[\underline{T}_2][\underline{T}_1]$ are both unit matrices, but:

$$[\underline{T}_2][\underline{T}_1] \neq [\underline{I}], \quad [\underline{T}_2][\underline{T}_1] = [\underline{I}]$$

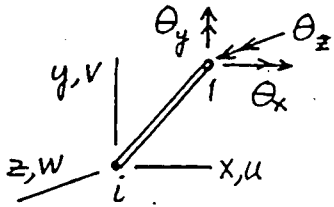
$\begin{matrix} 8 \times 6 & 6 \times 8 & 8 \times 8 & 6 \times 8 & 8 \times 6 & 6 \times 6 \end{matrix}$

The operation $[\underline{T}_1]\{\underline{d}_2\}$ expands 6 pieces of information to 8, and $[\underline{T}_2][\underline{T}_1]\{\underline{d}_2\}$ back again; nothing is lost. The operation $[\underline{T}_1][\underline{T}_2]\{\underline{d}_1\}$ condenses, then expands, but in expanding does not separate the v_i from their combination in the definition of the θ_i .

8.5-5

$$\begin{array}{c}
 \begin{array}{c} \uparrow v \\ \curvearrowright \theta \\ \uparrow u \end{array}
 \end{array}
 \begin{bmatrix}
 u_i & w_i & \theta_i \\
 u_j & w_j & \theta_j
 \end{bmatrix}
 =
 \begin{bmatrix}
 1 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & 1
 \end{bmatrix}
 \begin{bmatrix}
 u_1 & w_1 & \theta_1 \\
 u_2 & w_2 & \theta_2
 \end{bmatrix}
 \begin{bmatrix}
 b_1 & -a_1 & 1 \\
 b_2 & -a_2 & 1
 \end{bmatrix}$$

8.5-6



Link $j2$ is similar.

$$[u_i \ v_i \ w_i \ \theta_{xi} \ \theta_{yi} \ \theta_{zi} \ u_j \ v_j \ w_j \ \theta_{xj} \ \theta_{yj} \ \theta_{zj}]^T =$$

$$\begin{bmatrix} \tilde{T}_1 & \underline{0} \\ \underline{0} & \tilde{T}_2 \end{bmatrix}_{12 \times 12} \begin{bmatrix} u_1 \ v_1 \ w_1 \ \theta_{x1} \ \theta_{y1} \ \theta_{z1} \\ u_2 \ v_2 \ w_2 \ \theta_{x2} \ \theta_{y2} \ \theta_{z2} \end{bmatrix}^T$$

$$[I_1]_{6 \times 6} = \begin{bmatrix} 1 & 0 & 0 & 0 & -(z_1 - z_i) & (y_1 - y_i) \\ 0 & 1 & 0 & (z_1 - z_i) & 0 & -(x_1 - x_i) \\ 0 & 0 & 1 & -(y_1 - y_i) & (x_1 - x_i) & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

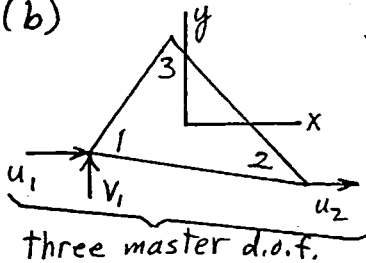
Obtain $[I_2]$ from $[I_1]$ by changing i to j and 1 to 2 .

8.5-7

(a) $\theta = \frac{u_3 - u_2}{b}$ $v_1 = v_3 + \theta a$
 $v_2 = v_3$
 $u_1 = u_3$

$$\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -a/b & a/b & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

(b)



$\theta = \frac{u_2 - u_1}{y_1 - y_2}$

(OK if $y_1 \neq y_2$)

Relative to node 1,

$v_2 = \theta(x_2 - x_1)$

$u_3 = -\theta(y_3 - y_1)$

$v_3 = \theta(x_3 - x_1)$

Let $\alpha = \frac{1}{y_1 - y_2}$

$$\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(x_2 - x_1)\alpha & 1 & (x_2 - x_1)\alpha \\ 1 + (y_3 - y_1)\alpha & 0 & -(y_3 - y_1)\alpha \\ -(x_3 - x_1)\alpha & 1 & (x_3 - x_1)\alpha \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \end{Bmatrix}$$

8.6-1

Along AB Apply $\{\underline{\sigma}\} = [\underline{E}]\{\underline{\epsilon}\}$
with $\epsilon_x = u_{,x}$, etc.

$$\sigma_x = -p, \text{ so } -p = \frac{E}{1-\nu^2} (u_{,x} + \nu v_{,y})$$

$$\tau_{xy} = 0, \text{ so } 0 = u_{,y} + v_{,x}$$

These are 2 constraint eqs. on 4 d.o.f.

Along CD Same as AB, except $p = 0$.

Along AD Can transform d.o.f. to the st axes by use of Eqs. 8.1-3 & 8.2-3; then treat like edge CD.

Along BC $u = v = u_{,x} = v_{,x} = 0$. But $u_{,y}$ & $v_{,y}$ are unknown, as are corresp. terms in $\{\underline{R}\}$. Can perhaps set these load terms to zero with little consequence

Anisotropy E.g. along AB,

$$\begin{Bmatrix} -p \\ \sigma_y \\ 0 \end{Bmatrix} = [\underline{E}] \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad \text{From the 1st \& 3rd of these equations,}$$

$$\begin{aligned} -p &= E_{11}u_{,x} + E_{12}v_{,y} + E_{13}(u_{,y} + v_{,x}) \\ 0 &= E_{31}u_{,x} + E_{32}v_{,y} + E_{33}(u_{,y} + v_{,x}) \end{aligned} \quad \left. \vphantom{\begin{aligned} -p &= E_{11}u_{,x} + E_{12}v_{,y} + E_{13}(u_{,y} + v_{,x}) \\ 0 &= E_{31}u_{,x} + E_{32}v_{,y} + E_{33}(u_{,y} + v_{,x}) \end{aligned}} \right\} \begin{array}{l} 2 \text{ con-} \\ \text{straint} \\ \text{eqs.} \end{array}$$

8.6-2

(a) Use the beam N_i of Fig. 3.2-4:

$$u = [N_1 \ N_2 \ N_3 \ N_4] [u_1 \ \epsilon_{x1} \ u_2 \ \epsilon_{x2}]^T$$

$[B] = \frac{d}{dx} [N]$, etc. However, it is easier to use the "a-basis", $u = [1 \ x \ x^2 \ x^3] \{a\}$

$$[k_a] = \int_0^L [B_a]^T [B_a] AE dx, \text{ where } [B_a] =$$

$$[0 \ 1 \ 2x \ 3x^2], [k] = [A]^{-T} [k_a] [A]^{-1}$$

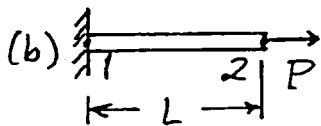
and $[A]^{-1}$ is computed in Prob. 3.2-4

$$AE \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & L & L^2 & L^3 \\ 0 & L^2 & \frac{4}{3}L^3 & \frac{3}{2}L^4 \\ 0 & L^3 & \frac{3}{2}L^4 & \frac{9}{5}L^5 \end{bmatrix}, AE \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -L & -\frac{L^2}{6} & L & \frac{L^2}{6} \\ -9L^2 & -2L^3 & 9L^2 & 3L^3 \end{bmatrix}$$

$[k_a] \qquad [k_a] [A]^{-1}$

$$\begin{bmatrix} 1 & 0 & -3/L^2 & 2/L^3 \\ 0 & 1 & -2/L & 1/L^2 \\ 0 & 0 & 3/L^2 & -2/L^3 \\ 0 & 0 & -1/L & 1/L^2 \end{bmatrix} \frac{AE}{30L} \begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix}$$

$[A]^{-T} \qquad [k]$



$u_1 = 0$, so $\frac{AE}{30L} \begin{bmatrix} 4L^2 & -3L & -L^2 \\ -3L & 36 & -3L \\ -L^2 & -3L & 4L^2 \end{bmatrix} \begin{Bmatrix} \epsilon_{x1} \\ u_2 \\ \epsilon_{x2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \\ F_{x2} \end{Bmatrix}$

now impose $\epsilon_{x2} = \frac{P}{AE}$: $\frac{AE}{30L} \begin{bmatrix} 4L^2 & -3L & 0 \\ -3L & 36 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \epsilon_{x1} \\ u_2 \\ \epsilon_{x2} \end{Bmatrix} = \begin{Bmatrix} (AE/30L)L^2(P/AE) \\ P + (AE/30L)3L(P/AE) \\ P/AE \end{Bmatrix}$

from the first two equations, inverting the 2 by 2 matrix,

$$\begin{Bmatrix} \epsilon_{x1} \\ u_2 \end{Bmatrix} = \frac{30L}{AE} \frac{1}{135L^2} \begin{bmatrix} 36 & 3L \\ 3L & 4L^2 \end{bmatrix} \begin{Bmatrix} PL/30 \\ 1.1P \end{Bmatrix} = \frac{1}{4.5AEL} \begin{Bmatrix} 1.2PL + 3.3PL \\ 0.1PL^2 + 4.4PL^2 \end{Bmatrix}$$

$$\begin{Bmatrix} \epsilon_{x1} \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P/AE \\ PL/AE \end{Bmatrix} \quad \checkmark$$

8.7-1

$$J = [N, \xi] \{x\}$$

$$J = \begin{bmatrix} -\frac{1+2\xi}{2} & -2\xi & \frac{1+2\xi}{2} \end{bmatrix} \begin{Bmatrix} 0 \\ L/3 \\ L \end{Bmatrix} = \frac{L}{3} \left(\frac{3}{2} + \xi \right)$$

$$J = 0 \text{ at } \xi = -\frac{3}{2}$$

$$x = [N] \{x\} = \begin{bmatrix} -\frac{\xi+\xi^2}{2} & 1-\xi^2 & \frac{\xi+\xi^2}{2} \end{bmatrix} \begin{Bmatrix} 0 \\ L/3 \\ L \end{Bmatrix} = \frac{L}{3} \left(1 + \frac{3}{2}\xi + \frac{1}{2}\xi^2 \right)$$

$$\text{For } \xi = -\frac{3}{2}, \quad x = \frac{L}{3} \left(1 - \frac{9}{4} + \frac{9}{8} \right) = -\frac{L}{24}, \text{ so } \frac{x}{L} = -0.04167$$

(a) From Eqs. 3.2-3 and 3.2-8,

$$\underline{v} = [\underline{X}][\underline{A}]^{-1}\{\underline{d}\} \quad \text{Hence}$$

$$[\underline{k}_f] = \beta [\underline{A}]^{-T} \int_0^L \underbrace{\begin{pmatrix} 1 \\ x \\ x^2 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix}^T}_{b} dx [\underline{A}]^{-1}$$

$b = \text{width of the beam}$

$$\begin{bmatrix} L & L^2/2 & L^3/3 & L^4/4 \\ L^3/3 & L^4/4 & L^5/5 & \\ \text{symm.} & L^5/5 & L^6/6 & \\ & & L^7/7 & \end{bmatrix} \underline{b} = [\underline{g}]. \quad \text{Hence,}$$

$$[\underline{k}_f] = \beta [\underline{A}]^{-T} [\underline{g}] [\underline{A}]^{-1} =$$

$$\frac{\beta b L}{420} \begin{bmatrix} 156 & 22L & 54 & -13L \\ & 4L^2 & 13L & -3L^2 \\ \text{symm.} & & 156 & -22L \\ & & & 4L^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$(b) \underline{v} = \underbrace{\begin{bmatrix} \frac{L-x}{L} & \frac{(L-x)x}{2L} & \frac{x}{L} & -\frac{(L-x)x}{2L} \end{bmatrix}}_{[\underline{N}]} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$[\underline{k}_f] = \beta \int_0^L [\underline{N}]^T [\underline{N}] b dx \quad \text{After tedious expansion \& integration,}$$

$$[\underline{k}_f] = \frac{\beta b L}{120} \begin{bmatrix} 40 & 5L & 20 & -5L \\ & L^2 & 5L & -L^2 \\ \text{symm.} & & 40 & -5L \\ & & & L^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$(c) \underline{v} = \begin{bmatrix} \frac{w_1}{L} & \frac{w_2}{L} \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} = [\underline{N}] \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix}$$

$$[\underline{k}_f] = \beta \int_0^L [\underline{N}]^T [\underline{N}] b dx = \frac{\beta b L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix}$$

(Can include θ_1 & θ_2 by adding zeros to $[\underline{k}_f]$)

(d) Rigid body motion; gives diagonal $[\underline{k}_f]$.

$$\text{For } \{\underline{d}\} = [v_1, \theta_1, v_2, \theta_2]^T,$$

$$[\underline{k}_f] = \frac{\beta b L}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$$

All these formulations are valid; that is, all provide correct convergence with mesh refinement

8.8-2

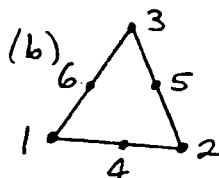
In area coords.,



$$w = \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}$$

$$[k_f] = \beta \int_A \begin{bmatrix} \xi_1^2 & \xi_1 \xi_2 & \xi_1 \xi_3 \\ \xi_1 \xi_2 & \xi_2^2 & \xi_2 \xi_3 \\ \xi_1 \xi_3 & \xi_2 \xi_3 & \xi_3^2 \end{bmatrix} dA$$

Integrate by use of Eq. 7.3-7.



$$[k_f] = \frac{\beta A}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$w = [N] \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{Bmatrix}$$

N_i given by Eq. 7.3-4.

Integrate by use of Eq. 7.3-7.

For $i=1,2,3$, $\int_A N_i^2 dA = \frac{A}{30}$

For i or $j=1,2,3$ but $i \neq j$, $\int_A N_i N_j dA = -\frac{A}{180}$

For $i=4,5,6$, $\int_A N_i^2 dA = \frac{8A}{45}$

For i or $j=4,5,6$ but $i \neq j$, $\int_A N_i N_j dA = \frac{4A}{45}$

$\int_A N_1 N_5 dA = \int_A N_2 N_6 dA = \int_A N_3 N_4 dA = -\frac{A}{45}$

Integrals of $N_1 N_4$, $N_1 N_6$, $N_2 N_4$, $N_2 N_5$, $N_3 N_5$, and $N_3 N_6$ are zero. Hence

$$[k_f] = \frac{\beta A}{180} \begin{bmatrix} 6 & -1 & -1 & 0 & -4 & 0 \\ -1 & 6 & -1 & 0 & 0 & -4 \\ 1 & -1 & 6 & -4 & 0 & 0 \\ 0 & 0 & -4 & 32 & 16 & 16 \\ -4 & 0 & 0 & 16 & 32 & 16 \\ 0 & -4 & 0 & 16 & 16 & 32 \end{bmatrix}$$

8.8-3

Strain energy per unit of area A is

$$dU = \frac{\beta}{2} w dA (w) + \frac{\alpha}{2} w_{,x} dA (w_{,x}) + \frac{\alpha}{2} w_{,y} dA (w_{,y})$$

where $\beta w dA = \text{force}$ $\alpha w_{,x} dA = \text{moment}$
 $w = \text{deflection}$ $w_{,x} = \text{rotation}$

Hence

$$U = \frac{1}{2} \int_A [w \ w_{,x} \ w_{,y}] \begin{bmatrix} \beta & & \\ & \alpha & \\ & & \alpha \end{bmatrix} \begin{Bmatrix} w \\ w_{,x} \\ w_{,y} \end{Bmatrix} dA$$

Let $[w \ w_{,x} \ w_{,y}]^T = [\underline{Q}] \{ \underline{d} \}$. Then
 $3 \times n \quad n \times 1$

$$[\underline{k}_s] = \int_A [\underline{Q}]^T [\beta \ \alpha \ \alpha] [\underline{Q}] dA$$

$n \times n$

8.8-4

(a) Subs. $\xi = 1 - \frac{2a}{r}$ into $\phi = \frac{1}{2}(1-\xi)\phi_1 + \frac{1}{2}(1+\xi)\phi_3$

$$\phi = \left(\frac{1}{2} - \frac{1}{2} + \frac{a}{r}\right)\phi_1 + \left(\frac{1}{2} + \frac{1}{2} - \frac{a}{r}\right)\phi_3 = \frac{a}{r}\phi_1 + \left(1 - \frac{a}{r}\right)\phi_3$$

$\phi = c$ if $\phi_1 = \phi_3 = c$, $\phi \rightarrow \phi_3$ as $r \rightarrow \infty$

(b) From Eq. 8.8-3,

$$J = \frac{\partial x}{\partial \xi} = \frac{(1-\xi)(-2) - (-2\xi)(-1)}{(1-\xi)^2} x_1 + \frac{(1-\xi)(1) - (1+\xi)(-1)}{(1-\xi)^2} x_2$$

$$J = \frac{-2}{(1-\xi)^2} x_1 + \frac{2}{(1-\xi)^2} x_2 = \frac{2(x_2 - x_1)}{(1-\xi)^2} = \frac{2a}{(1-\xi)^2}$$

$$(c) [B] = \frac{1}{J} \frac{d}{d\xi} \begin{bmatrix} 1-\xi & 1+\xi \\ 2 & 2 \end{bmatrix} = \frac{(1-\xi)^2}{4a} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$[k] = \int_{-1}^1 [B]^T E [B] J A d\xi = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \frac{EA}{16a^2} 2a \int_{-1}^1 (1-\xi)^2 d\xi$$

$$[k] = \frac{EA}{8a} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \frac{8}{3} = \frac{EA}{3a} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(a)

$$\phi = [N] \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_5 \\ \phi_6 \end{Bmatrix}$$

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2 = \frac{1}{4}(1-\xi)(1+\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_4 = \frac{1}{4}(1+\xi)(1+\eta)$$

Mapping functions

 N_i same as in

Fig. 8.8-4.

(b) Using derivatives of N_i in Fig. 8.8-4,

$$[J] = \begin{bmatrix} \frac{-2}{(1-\xi)^2} \frac{1-\eta}{2} & \frac{-2}{(1-\xi)^2} \frac{1+\eta}{2} & \frac{2}{(1-\xi)^2} \frac{1-\eta}{2} & \frac{2}{(1-\xi)^2} \frac{1+\eta}{2} \\ \frac{\xi}{1-\xi} & \frac{-\xi}{1-\xi} & -\frac{1+\xi}{2(1-\xi)} & \frac{1+\xi}{2(1-\xi)} \end{bmatrix} \begin{bmatrix} a & 0 \\ a & 2b \\ 2a & 0 \\ 2a & 2b \end{bmatrix}$$

$$[J] = \begin{bmatrix} \frac{a}{(1-\xi)^2} & 0 \\ 0 & b \end{bmatrix}, J = \frac{ab}{(1-\xi)^2}$$

$$(c) [\underline{B}] = \begin{bmatrix} \underline{N}_{,x} \\ \underline{N}_{,y} \end{bmatrix} = [J]^{-1} \begin{bmatrix} \underline{N}_{,\xi} \\ \underline{N}_{,\eta} \end{bmatrix} \quad \text{with only } \phi_1 \text{ \& } \phi_2 \text{ active.}$$

$$[\underline{B}] = \frac{(1-\xi)^2}{ab} \begin{bmatrix} b & 0 \\ 0 & \frac{a}{(1-\xi)^2} \end{bmatrix} \frac{1}{4} \begin{bmatrix} -(1-\eta) & -(1+\eta) \\ -(1-\xi) & (1-\xi) \end{bmatrix}$$

$$[\underline{B}] = \frac{(1-\xi)^2}{4ab} \begin{bmatrix} -b(1-\eta) & -b(1+\eta) \\ -\frac{a}{1-\xi} & \frac{a}{1-\xi} \end{bmatrix}$$

$$[\underline{k}] = \int_{-1}^1 \int_{-1}^1 [\underline{B}]^T k [\underline{B}] t J d\xi d\eta$$

$$[\underline{k}] = \frac{kt}{16ab} \int_{-1}^1 \int_{-1}^1 \begin{bmatrix} b^2(1-\xi)^2(1-\eta)^2 + a^2 \\ b^2(1-\xi)^2(1-\eta^2) - a^2 \\ b^2(1-\xi)^2(1-\eta^2) - a^2 \\ b^2(1-\xi)^2(1+\eta)^2 + a^2 \end{bmatrix} d\xi d\eta$$

$$\int_{-1}^1 (1+\eta^2) d\eta = \frac{8}{3}$$

$$\int_{-1}^1 (1-\eta^2) d\eta = \frac{4}{3}$$

$$[\underline{k}] = \frac{kt}{16ab} \begin{bmatrix} \frac{64}{9}b^2 + 2a^2 & \frac{32}{9}b^2 - 2a^2 \\ \frac{32}{9}b^2 - 2a^2 & \frac{64}{9}b^2 + 2a^2 \end{bmatrix}$$

8.8-6

$$M_1 = -\frac{2\xi}{1-\xi} \frac{-\eta+\eta^2}{2} \quad M_4 = \frac{1+\xi}{1-\xi} \frac{-\eta+\eta^2}{2}$$

$$M_2 = -\frac{2\xi}{1-\xi} (1-\eta^2) \quad M_5 = \frac{1+\xi}{1-\xi} (1-\eta^2)$$

$$M_3 = -\frac{2\xi}{1-\xi} \frac{\eta+\eta^2}{2} \quad M_6 = \frac{1+\xi}{1-\xi} \frac{\eta+\eta^2}{2}$$

8.9-1

$$(a) K^* = K + \Delta K, \Delta K = K^* - K = 0.3$$

$$\text{Iterative eq. is } 0.5 D_{i+1}^* = 2 - 0.3 D_i^* \\ \text{i.e. } D_{i+1}^* = 4 - 0.6 D_i^*$$

$$\begin{aligned} D_2^* &= 4 - 0.6(4) = 1.6 \\ D_3^* &= 4 - 0.6(1.6) = 3.04 \\ D_4^* &= 4 - 0.6(3.04) = 2.176 \\ D_5^* &= 4 - 0.6(2.176) = 2.6944 \\ D_6^* &= 4 - 0.6(2.6944) = 2.38336 \end{aligned} \left. \begin{array}{l} \text{Converges} \\ \text{to } D^* = \\ 2.50, \text{ i.e.} \\ \text{to exact} \\ \text{value.} \end{array} \right\}$$

$$(b) D_{i+1}^* = \frac{1}{k} (R - \Delta k D_i^*) = D - \frac{\Delta k}{k} D_i^*$$

$$D_2^* = D - \frac{\Delta k}{k} D$$

$$D_3^* = D - \frac{\Delta k}{k} D_1^* = D - \frac{\Delta k}{k} D + \left(\frac{\Delta k}{k}\right)^2 D$$

$$D_4^* = D - \frac{\Delta k}{k} D_2^* = D \left[1 - \frac{\Delta k}{k} + \left(\frac{\Delta k}{k}\right)^2 - \left(\frac{\Delta k}{k}\right)^3 \right]$$

etc., so the series in brackets extends.

The series converges if $\left| \frac{\Delta k}{k} \right| < 1$.

Hence, magnitude of increase or decrease in k must be less than 100%.

9.2-1

Exact solution:

$$\begin{bmatrix} 1.00 & 1.00 \\ 1.00 & 1.01 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 2.00 \\ 2.01 \end{Bmatrix}, \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 101 & -100 \\ -100 & 100 \end{bmatrix} \begin{Bmatrix} 2.00 \\ 2.01 \end{Bmatrix}$$

from which $x = y = 1$.

Small change in matrix: e.g.,

$$\begin{bmatrix} 1.00 & 1.00 \\ 1.00 & 1.02 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 2.00 \\ 2.01 \end{Bmatrix}, \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 51 & -50 \\ -50 & 50 \end{bmatrix} \begin{Bmatrix} 2.00 \\ 2.01 \end{Bmatrix}$$

from which $x = 1.5$, $y = 0.5$.

(Or, if 1.01 changed to 1.00, the matrix is singular & the equations contradictory.)

Small change in vector: e.g.,

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 101 & -100 \\ -100 & 100 \end{bmatrix} \begin{Bmatrix} 2.00 \\ 2.02 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 2 \end{Bmatrix}$$

9.2-2

multiply 1st eq. by $\frac{\alpha cs}{1 + \alpha c^2}$.

$$\begin{bmatrix} k\alpha cs & \frac{k(\alpha cs)^2}{1 + \alpha c^2} \\ k\alpha cs & k(1 + \alpha s^2) \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} = \begin{Bmatrix} \frac{\alpha cs}{1 + \alpha c^2} P \\ 0 \end{Bmatrix}$$

Subtract 1st eq. from 2nd. 2nd eq. becomes

$$k \left[(1 + \alpha s^2) - \frac{(\alpha cs)^2}{1 + \alpha c^2} \right] v_1 = -\frac{\alpha cs}{1 + \alpha c^2} P = -\frac{cs P}{\left(\frac{1}{\alpha} + c^2\right)}$$

As α becomes large, we approach

$$k[\alpha s^2 - \alpha s^2] v_1 = -\frac{s}{c} P$$

↖ severe cancellation error

9.2-3

$$EI \begin{bmatrix} \frac{12}{L^3} + \frac{12}{(\alpha L)^3} & -\frac{6}{L^2} + \frac{6}{(\alpha L)^2} & \frac{6}{(\alpha L)^2} \\ -\frac{6}{L^2} + \frac{6}{(\alpha L)^2} & \frac{4}{L} + \frac{4}{\alpha L} & \frac{2}{\alpha L} \\ \frac{6}{(\alpha L)^2} & \frac{2}{\alpha L} & \frac{4}{\alpha L} \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ M_0 \end{Bmatrix}$$

As α becomes small, we approach

$$EI \begin{bmatrix} \frac{12}{(\alpha L)^3} & \frac{6}{(\alpha L)^2} & \frac{6}{(\alpha L)^2} \\ \frac{6}{(\alpha L)^2} & \frac{4}{\alpha L} & \frac{2}{\alpha L} \\ \frac{6}{(\alpha L)^2} & \frac{2}{\alpha L} & \frac{4}{\alpha L} \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ M_0 \end{Bmatrix}$$

$(\alpha L \text{ times row 1}) - (\text{row 2}) = (\text{row 3})$, i.e.
the matrix becomes singular.

9.2-4

$$(a) \frac{EI}{L^3} \begin{bmatrix} 12 + \frac{kL^3}{EI} & -12 & 6L \\ -12 & 12 & -6L \\ 6L & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ 0 \end{Bmatrix}$$

Add $\frac{3}{2L}$ times last eq. to 2nd eq.; subtract $\frac{3}{2L}$ times last eq. from 1st eq.; finally divide last eq. by its pivot. Thus

$$\frac{EI}{L^3} \begin{bmatrix} 3 + \frac{kL^3}{EI} & -3 & 0 \\ -3 & 3 & 0 \\ \frac{3}{2L} & -\frac{3}{2L} & 1 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ 0 \end{Bmatrix}$$

Add 2nd eq. to 1st, then divide 2nd by 3.

$$\frac{EI}{L^3} \begin{bmatrix} kL^3/EI & 0 & 0 \\ -1 & 1 & 0 \\ \frac{3}{2L} & -\frac{3}{2L} & 1 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} -P \\ -P/3 \\ 0 \end{Bmatrix}$$

$$w_1 = -P/k$$

$$w_2 = w_1 - \frac{PL^3}{3EI} = -P \left(\frac{1}{k} + \frac{L^3}{3EI} \right)$$

$$\theta_2 = \frac{3}{2L} (-w_1 + w_2) = -\frac{PL^2}{2EI}$$

But lead coef. in last matrix eq. obtained as $(3 + \frac{kL^3}{EI}) - 3$; severe cancellation error if k is small.

(b)  Set $\theta_2 = \theta_1 + \theta_{21}$
& $w_2 = w_1 + L\theta_1 + w_{21}$

$$\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & 1 & 0 \\ & & & 1 \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_{21} \\ \theta_{21} \end{Bmatrix} = [\underline{T}] \{ \underline{d} \}$$

Applied to $[k]$ of standard beam el.,

$[\underline{T}]^T [k] [\underline{T}]$ yields

$$\frac{EI}{L^3} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & -6L \\ 0 & 0 & -6L & 4L^2 \end{bmatrix}$$

Add stiffness k to w_1 & set $\theta_1 = 0$. Thus

$$\frac{EI}{L^3} \begin{bmatrix} kL^3/EI & 0 & 0 \\ 0 & 12 & -6L \\ 0 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_{21} \\ \theta_{21} \end{Bmatrix} = \begin{Bmatrix} -P \\ -P \\ 0 \end{Bmatrix}$$

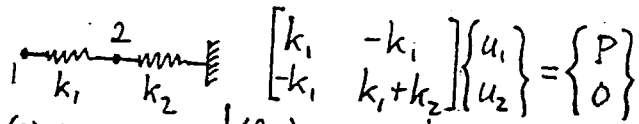
where the load vector is $[\underline{T}]^T [0, 0, -P, 0]^T$ with the second term discarded for $\theta_1 = 0$. The condition of the matrix is independent of k . The solution is

$$\begin{Bmatrix} w_1 \\ w_{21} \\ \theta_{21} \end{Bmatrix} = \begin{Bmatrix} -P/k \\ -PL^3/3EI \\ -PL^2/2EI \end{Bmatrix}, \text{ hence}$$

$$w_2 = -\frac{P}{k} + L(0) + w_{21} = -\frac{P}{k} - \frac{PL^3}{3EI}$$

$$\theta_2 = (0) + \theta_{21} = -\frac{PL^2}{2EI}$$

9.3-1



(a) Unscaled: $\begin{vmatrix} 10-\lambda & -10 \\ -10 & 11-\lambda \end{vmatrix} = \lambda^2 - 21\lambda + 10 = 0$

$C(K) = \frac{20.51}{0.487} = 42.07$

Scaled: $\begin{vmatrix} 10-10\lambda & -10 \\ -10 & 11-11\lambda \end{vmatrix} = 110\lambda^2 - 220\lambda + 10 = 0$

$C(K) = \frac{1.953}{0.0465} = 41.98$

(b) Unscaled: $\begin{vmatrix} 1-\lambda & -1 \\ -1 & 11-\lambda \end{vmatrix} = \lambda^2 - 12\lambda + 10 = 0$

$C(K) = \frac{11.10}{0.901} = 12.32$

Scaled: $\begin{vmatrix} 1-\lambda & -1 \\ -1 & 11-11\lambda \end{vmatrix} = 11\lambda^2 - 22\lambda + 10 = 0$

$C(K) = \frac{1.302}{0.698} = 1.863$

9.3-2



$$(a) [K] = k \begin{bmatrix} c & -c \\ -c & c+1 \end{bmatrix}, \quad \begin{vmatrix} c-\lambda & -c \\ -c & c+1-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - (2c+1)\lambda + c = 0, \quad \lambda = \frac{2c+1 \pm \sqrt{4c^2+1}}{2}$$

$$C(K) = \frac{2c+1 + (4c^2+1)^{1/2}}{2c+1 - (4c^2+1)^{1/2}}$$

$$\text{By trial, } C(K)_{\min} = 5.8284 \text{ for } c = 0.50$$

$$(b) \begin{vmatrix} c(1-\lambda) & -c \\ -c & (c+1)(1-\lambda) \end{vmatrix} = c(c+1)(1-\lambda)^2 - c^2 = 0$$

$$(1-\lambda)^2 = \frac{c}{c+1}, \quad \lambda = 1 \pm \left(\frac{c}{c+1}\right)^{1/2}$$

$$C(K) = \frac{1 + \left(\frac{c}{c+1}\right)^{1/2}}{1 - \left(\frac{c}{c+1}\right)^{1/2}}, \quad C(K)_{\min} = 1.0 \text{ for } c = 0$$

9.3-3

$\cos \beta = \sin \beta = \frac{\sqrt{2}}{2}$. D.o.f. are u_1, v_1 .

$$[K] = \alpha k \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} + k \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + k \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[K] = \frac{k}{2} \begin{bmatrix} 2+\alpha & \alpha \\ \alpha & 2+\alpha \end{bmatrix}$$

$$(a) \begin{vmatrix} 2+\alpha-\lambda & \alpha \\ \alpha & 2+\alpha-\lambda \end{vmatrix} = 0, (2+\alpha-\lambda)^2 - \alpha^2 = 0$$

$$2+\alpha-\lambda = \pm \alpha \quad \lambda_{max} = 2+2\alpha \quad C(K) = 1+\alpha$$

$$\lambda_{min} = 2$$

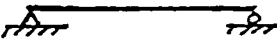
$$(b) \begin{vmatrix} (2+\alpha)-\lambda(2+\alpha) & \alpha \\ \alpha & (2+\alpha)-\lambda(2+\alpha) \end{vmatrix} = 0$$

$$[(1-\lambda)(2+\alpha)]^2 - \alpha^2 = 0, 1-\lambda = \pm \frac{\alpha}{2+\alpha}$$

$$C(K) = \frac{1 + \frac{\alpha}{2+\alpha}}{1 - \frac{\alpha}{2+\alpha}} = \frac{2+2\alpha}{2} = 1+\alpha$$

9.3-4

$$[k] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ \text{symm.} & & 12 & -6L \\ & & & 4L^2 \end{bmatrix} \begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{matrix}$$

(a)  (simply supported)

$$\begin{vmatrix} 4L^2(1-\lambda) & 2L^2 \\ 2L^2 & 4L^2(1-\lambda) \end{vmatrix} = 0, \begin{vmatrix} 2(1-\lambda) & 1 \\ 1 & 2(1-\lambda) \end{vmatrix} = 0$$

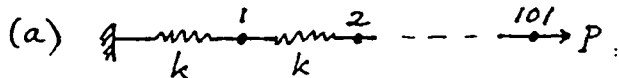
$$2(1-\lambda) = \pm 1, C(K) = \frac{3/2}{1/2} = 3$$

(b)  (cantilever)

$$\begin{vmatrix} 12(1-\lambda) & -6L \\ -6L & 4L^2(1-\lambda) \end{vmatrix} = 0, \begin{vmatrix} 6(1-\lambda) & -3 \\ -3 & 2(1-\lambda) \end{vmatrix} = 0$$

$$\sqrt{12}(1-\lambda) = \pm 3, C(K) = \frac{1+3/\sqrt{12}}{1-3/\sqrt{12}} = 13.93$$

9.4-1



$$\begin{bmatrix} 200 & -100 & 0 & \dots \\ -100 & 200 & -100 & \dots \\ 0 & -100 & 200 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

← NW corner of $[K]$.
Add $0.5 \times$ row 1 to row 2 & normalize row 1.

$$\begin{bmatrix} 1 & -0.5 & 0 & \dots \\ 0 & 150 & -100 & \dots \\ 0 & -100 & 200 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Next add $\frac{100}{150} \times$ row 2 to row 3 & normalize row 2.

$$\begin{bmatrix} 1 & 0.5 & 0 & 0 & \dots \\ 0 & 1 & -0.667 & 0 & \dots \\ 0 & 0 & 133.3 & -100 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

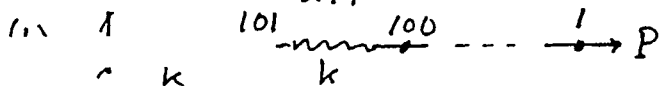
In general, after $n-1$ eliminations, the n^{th} diagonal coef. becomes $100 + \frac{100}{n}$. Thus, after $n-1 = 99$ eliminations,

$$\begin{bmatrix} 101 & -100 \\ -100 & 100 \end{bmatrix} \begin{Bmatrix} u_{100} \\ u_{101} \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix} \quad \text{Decay ratio is } \frac{200}{101} = 1.98$$

$$\begin{bmatrix} 100 & \frac{-100}{101} 100 \\ -100 & 100 \end{bmatrix} \begin{Bmatrix} u_{100} \\ u_{101} \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}$$

Last elimination gives $100 + \frac{-100}{101} 100 = 0.99$

Decay ratio is $\frac{100}{0.99} = 101$



$$\begin{bmatrix} 100 & -100 & 0 & \dots \\ -100 & 200 & -100 & \dots \\ 0 & -100 & 200 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

← NW corner of $[K]$.
Elimination changes 200's to 100's.

After 99 eliminations,

$$\begin{bmatrix} 100 & -100 \\ -100 & 200 \end{bmatrix} \begin{Bmatrix} u_{100} \\ u_{101} \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix} \quad \text{Decay ratio is } \frac{200}{100} = 2$$

Last elimination yields $100 u_{101} = P$

Again, decay ratio = $\frac{200}{100} = 2$.

9.4-2

- (a) One rigid body motion possible: last eq. (regardless of numbering sequence for the nodes).
- (b) Two rigid body motions possible: next to last eq. (a v d.o.f., in usual ordering).
- (c) Three rigid body motions possible: third from last eq. (a u d.o.f., in usual ordering).
- (d) One rigid body motion possible (axial translation: last equation).
- (e) Three rigid body motions possible. Let $\{D\} = [u_1 \ v_1 \ u_2 \ \dots \ u_n \ v_n]^T$. Trouble in third from last (v_{n-1}) provided that v_{n-1} and v_n are not collinear. If they are collinear, we need u_{n-1} to define rotation; then the trouble is detected in 4th from last eq.

9.4-3

From Prob. 9.2-2, original K_{zz} is $k(1+\alpha s^2)$ and reduced K_{zz} is $k(1+\alpha s^2) - \frac{k(\alpha cs)^2}{1+\alpha c^2} = k \frac{1+\alpha s^2+\alpha c^2}{1+\alpha c^2} = k \frac{1+\alpha}{1+\alpha c^2}$

Decay ratio is

$$\frac{(1+\alpha s^2)(1+\alpha c^2)}{1+\alpha} = \frac{1+\alpha+\alpha^2 c^2 s^2}{1+\alpha} = \frac{\frac{1}{\alpha} + 1 + \alpha c^2 s^2}{\frac{1}{\alpha} + 1}$$

Becomes large if α is large unless $c=0$ (i.e. $\beta = \frac{\pi}{2}$) or $s=0$ (i.e. $\beta=0$).

9.4-4

$$\text{Exact: } \begin{cases} u_2 \\ u_3 \end{cases} = \begin{cases} 0.30303030 \\ 0.30315530 \end{cases}$$

$$\begin{bmatrix} 8007 & -8000 \\ -8000 & 8000 \end{bmatrix} \begin{cases} u_2 \\ u_3 \end{cases} = \begin{cases} 1 \\ 1 \end{cases} \quad \begin{array}{l} \text{Solve by Gauss} \\ \text{elim. 4 digits} \end{array}$$

$$\begin{bmatrix} 1 & -0.9991 \\ 0 & 7.000 \end{bmatrix} \begin{cases} u_2 \\ u_3 \end{cases} = \begin{cases} 0.0001249 \\ 1.999 \end{cases}$$

$$\leftarrow 8000 - 8000(0.9991) = 8000 - 7993 = 7$$

$$\text{Diag. decay ratio} = \frac{8000}{7} \approx 10^3$$

$$\begin{cases} u_2 \\ u_3 \end{cases} = \begin{cases} 0.2854 \\ 0.2856 \end{cases} \quad \text{lost accuracy in 3 places}$$

$$\begin{vmatrix} 8007(1-\lambda) & -8000 \\ -8000 & 8000(1-\lambda) \end{vmatrix} = 0$$

$$8007(8000)(1-\lambda)^2 = 8000^2$$

$$8007(1-\lambda) = \pm 8000$$

$$C(K) = \frac{\lambda_{\max}}{\lambda_{\min}} = \frac{1 + \frac{8000}{8007}}{1 - \frac{8000}{8007}} = \frac{16,007}{7} = 2287$$

$$\log_{10} C(K) = 3.36 \approx \text{digits lost}$$

9.5-1

$$\{\Delta \underline{R}\} = \begin{Bmatrix} 2.88 \\ 1.52 \end{Bmatrix} - \begin{bmatrix} 1.78 & 1.06 \\ 0.94 & 0.56 \end{bmatrix} \begin{Bmatrix} 1.88 \\ -0.44 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -0.0008 \end{Bmatrix}$$

$$e = \frac{-0.44(-0.0008)}{1.88(2.88) + (-0.44)(1.52)} = 7.42(10)^{-5}$$

Exact:

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{0.0004} \begin{bmatrix} 0.56 & -1.06 \\ -0.94 & 1.78 \end{bmatrix} \begin{Bmatrix} 2.88 \\ 1.52 \end{Bmatrix} = \begin{Bmatrix} 4.0000 \\ -4.0000 \end{Bmatrix}$$

$\{\Delta \underline{R}\}$ and e are small, but error in solution is large. Equations are ill conditioned; they are the same as

$$1.78u_1 + 1.06u_2 = 2.88$$

$$1.78u_1 + 1.060426u_2 = 2.87830$$

9.5-2

Exact solution is $u_1 = u_2 = 1.0000$.

1st approx.:

$$\{\Delta R\} = \begin{Bmatrix} 2.0000 \\ 2.0001 \end{Bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix} \begin{Bmatrix} 2.0 \\ 0.0 \end{Bmatrix}$$

$$\{\Delta R\} = \begin{Bmatrix} 0.0000 \\ 0.0001 \end{Bmatrix}, e = \frac{2(0) + 0(0.0001)}{2(2) + 2.0001(0)} = 0$$

2nd approx.:

$$\{\Delta R\} = \begin{Bmatrix} 2.0000 \\ 2.0001 \end{Bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix} \begin{Bmatrix} 1.1 \\ 1.1 \end{Bmatrix} = \begin{Bmatrix} -0.20 \\ -0.20 \end{Bmatrix},$$

$$e = \frac{1.1(-0.20) + 1.1(-0.20)}{1.1(2) + 1.1(2.0001)} = \frac{-0.4}{4.0001} = -0.100$$

The more exact approximation has the larger residual.

9.5-3

$$\Delta u_{i+1} = k^{-1}(R - ku_i) = 0.04 \overbrace{(0.5 - 28u_i)}^{\Delta R}$$

$$u_{i+1} = u_i + \Delta u_{i+1} \quad \text{Start with } u_0 = 0.02$$

$$\Delta R_1 = -0.060$$

$$\Delta R_2 = 0.0072$$

$$\Delta u_1 = -0.0024$$

$$\Delta u_2 = 0.000288$$

$$u_1 = 0.0176$$

$$u_2 = 0.01789$$

$$\Delta R_3 = -0.000864$$

Exact:

$$\Delta u_3 = -0.0000346$$

$$u = \frac{0.5}{28} = 0.017857$$

$$u_3 = 0.017853$$

9.5-4

Exact: $u_1 = 7, u_2 = 4$

(a) First get initial soln. $\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_1 = \begin{bmatrix} 8 & 5 \\ 5 & 5 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 8 \\ 5 \end{Bmatrix}$

Now iterate:

$$\begin{Bmatrix} \Delta R_1 \\ \Delta R_2 \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} - \begin{bmatrix} .3 & -.3 \\ -.3 & .5 \end{bmatrix} \begin{Bmatrix} 8 \\ 5 \end{Bmatrix} = \begin{Bmatrix} .1 \\ -.1 \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta u_1 \\ \Delta u_2 \end{Bmatrix}_2 = \begin{bmatrix} 8 & 5 \\ 5 & 5 \end{bmatrix} \begin{Bmatrix} .1 \\ -.1 \end{Bmatrix} = \begin{Bmatrix} .3 \\ 0 \end{Bmatrix}, \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_2 = \begin{Bmatrix} 8.3 \\ 5.0 \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta R_1 \\ \Delta R_2 \end{Bmatrix}_2 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} - \begin{bmatrix} .3 & -.3 \\ -.3 & .5 \end{bmatrix} \begin{Bmatrix} 8.3 \\ 5.0 \end{Bmatrix} = \begin{Bmatrix} .01 \\ -.01 \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta u_1 \\ \Delta u_2 \end{Bmatrix}_3 = \begin{bmatrix} 8 & 5 \\ 5 & 5 \end{bmatrix} \begin{Bmatrix} .01 \\ -.01 \end{Bmatrix} = \begin{Bmatrix} .03 \\ 0 \end{Bmatrix}, \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_3 = \begin{Bmatrix} 8.33 \\ 5.00 \end{Bmatrix}$$

Next cycle gives $\begin{Bmatrix} 8.333 \\ 5.000 \end{Bmatrix}$ Conv. toward wrong answer.

(b) Iterate from same initial solution.

$$\begin{Bmatrix} \Delta R_1 \\ \Delta R_2 \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} - \begin{bmatrix} 1/3 & -1/3 \\ -1/3 & 7/12 \end{bmatrix} \begin{Bmatrix} 8 \\ 5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -.25 \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta u_1 \\ \Delta u_2 \end{Bmatrix}_2 = \begin{bmatrix} 8 & 5 \\ 5 & 5 \end{bmatrix} \begin{Bmatrix} 0 \\ -.25 \end{Bmatrix} = \begin{Bmatrix} -1.25 \\ -1.25 \end{Bmatrix}, \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_2 = \begin{Bmatrix} 6.75 \\ 3.75 \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta R_1 \\ \Delta R_2 \end{Bmatrix}_2 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} - \begin{bmatrix} 1/3 & -1/3 \\ -1/3 & 7/12 \end{bmatrix} \begin{Bmatrix} 6.75 \\ 3.75 \end{Bmatrix} = \begin{Bmatrix} 0 \\ .0625 \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta u_1 \\ \Delta u_2 \end{Bmatrix}_3 = \begin{bmatrix} 8 & 5 \\ 5 & 5 \end{bmatrix} \begin{Bmatrix} 0 \\ .0625 \end{Bmatrix} = \begin{Bmatrix} .3125 \\ .3125 \end{Bmatrix}, \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_3 = \begin{Bmatrix} 7.0625 \\ 4.0625 \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta R_1 \\ \Delta R_2 \end{Bmatrix}_3 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} - \begin{bmatrix} 1/3 & -1/3 \\ -1/3 & 7/12 \end{bmatrix} \begin{Bmatrix} 7.0625 \\ 4.0625 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -.015625 \end{Bmatrix}$$

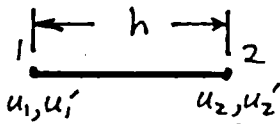
$$\begin{Bmatrix} \Delta u_1 \\ \Delta u_2 \end{Bmatrix}_4 = \begin{bmatrix} 8 & 5 \\ 5 & 5 \end{bmatrix} \begin{Bmatrix} 0 \\ -.015625 \end{Bmatrix} = \begin{Bmatrix} -.078125 \\ -.078125 \end{Bmatrix}$$

$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_4 = \begin{Bmatrix} 6.984375 \\ 3.984375 \end{Bmatrix}$ Conv. toward correct answer.

9.6-1

$$\text{At } x = L/2, \quad N_1 = N_3 = \frac{1}{2} \\ (L = h) \quad N_2 = \frac{L}{8}, \quad N_4 = -\frac{L}{8}$$

$$u = \frac{1}{2}(u_1 + u_2) + \frac{h}{8}(u'_1 - u'_2). \text{ Series:}$$



Expand in series, using $u, u', u'',$ etc. at middle.

$$u_1 = u - \frac{h}{2}u' + \frac{h^2}{8}u'' - \frac{h^3}{48}u''' + \frac{h^4}{384}u^{(4)} - \dots$$

$$u_2 = u + \frac{h}{2}u' + \frac{h^2}{8}u'' + \frac{h^3}{48}u''' + \frac{h^4}{384}u^{(4)} - \dots$$

$$u'_1 = u' - \frac{h}{2}u'' + \frac{h^2}{8}u''' - \frac{h^3}{48}u^{(4)} + \dots$$

$$u'_2 = u' + \frac{h}{2}u'' + \frac{h^2}{8}u''' + \frac{h^3}{48}u^{(4)} + \dots \quad \text{Hence}$$

$$u = \frac{1}{2}\left(2u + \frac{h^2}{4}u'' + \frac{h^4}{192}u^{(4)} + \dots\right)$$

$$+ \frac{h}{8}(-hu'' - \frac{h^3}{24}u^{(4)} - \dots) = u - \frac{h^4}{384}u^{(4)} + \dots$$

i.e. error term of order h^4 .

Similarly, error in u' is of order h^3 .

9.7-1

Full:

$$\frac{4.562(0.1768) - 7.124(0.3536)}{0.1768 - 0.3536} = 9.686$$

$$\frac{7.124(0.0884) - 8.302(0.1768)}{0.0884 - 0.1768} = 9.480$$

Reduced:

$$\frac{11.440(0.1768) - 9.323(0.3536)}{0.1768 - 0.3536} = 7.206$$

$$\frac{9.323(0.0884) - 8.922(0.1768)}{0.0884 - 0.1768} = 8.521$$

Hourglass Controlled:

$$\frac{8.572(0.1768) - 8.719(0.3536)}{0.1768 - 0.3536} = 8.866$$

$$\frac{8.719(0.0884) - 8.771(0.1768)}{0.0884 - 0.1768} = 8.823$$

9.7-2

$a + bx = y$. Substitute data values
and solve for a and b .

$$\begin{bmatrix} 1 & 1/\sqrt{52} \\ 1 & 1/\sqrt{94} \\ 1 & 1/\sqrt{130} \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{Bmatrix} 51.37 \\ 61.15 \\ 67.18 \end{Bmatrix} \text{ or } [Q]\{a\} = \{c\}$$

Form $[Q]^T[Q]\{a\} = [Q]^T\{c\}$; solve for $\{a\}$.

$$\begin{bmatrix} 3 & 0.3295229756 \\ \text{symm.} & 0.0375613748 \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{Bmatrix} 179.7000000 \\ 19.32295396 \end{Bmatrix}$$

we obtain $a = 93.3$, $b = -304.1$.

9.7-3

$$h_1 = \frac{1}{\sqrt{52}} = 0.1387, \sigma_1 = 51.37$$

$$h_2 = \frac{1}{\sqrt{94}} = 0.1031, \sigma_2 = 61.15$$

$$h_3 = \frac{1}{\sqrt{130}} = 0.0877, \sigma_3 = 67.18$$

Use Eq. 9.7-1:

$$\sigma_A = \frac{51.37(0.1031) - 61.15(0.1387)}{0.1031 - 0.1387} = 89.62$$

$$\sigma_B = \frac{61.15(0.0877) - 67.18(0.1031)}{0.0877 - 0.1031} = 101.52$$

$$\sigma_C = \frac{51.37(0.0877) - 67.18(0.1387)}{0.0877 - 0.1387} = 94.37$$

$$\frac{1}{3}(\sigma_A + \sigma_B + \sigma_C) = \frac{285.5}{3} = 95.17$$

9.7-4

$$h_i = \frac{1}{N_i^{1/2}} \quad h_1 = \frac{1}{10}, \quad h_2 = \frac{1}{10\sqrt{2}}, \quad h_3 = \frac{1}{20}$$

By trial, we discover that convergence rate is $O(h^4)$, e.g.

$$\frac{4.64 - 4.16}{h_1^4 - h_2^4} = 6400 \quad \text{and} \quad \frac{4.76 - 4.64}{h_2^4 - h_3^4} = 6400$$

$$\text{Eq. 9.7-1: } \phi^0 = \frac{4.76h_2^4 - 4.64h_3^4}{h_2^4 - h_3^4} = 4.80$$

9.7-5

(a) Fig. 3.5-2: $e = O(h^2)$ for displacement, $e = O(h)$ for stress

$$v_A = \frac{0.859(1^2) - 0.961(2^2)}{1^2 - 2^2} = 0.995, \quad \sigma_{xB} = \frac{0.854(1) - 0.956(2)}{1 - 2} = 1.058$$

(b) CST results in Fig. 3.10-2: $e = O(h^2)$ for displacement

$$N=2, N=4: v_A = \frac{0.502(1^2) - 0.765(2^2)}{1^2 - 2^2} = 0.853$$

$$N=4, N=8: v_A = \frac{0.765(1^2) - 0.921(2^2)}{1^2 - 2^2} = 0.973$$

(c) Ref. 3.9 results in Fig. 3.10-2: $e = O(h^2)$ for displacement

$$N=2, N=4: v_A = \frac{0.852(1^2) - 0.954(2^2)}{1^2 - 2^2} = 0.988$$

$$N=4, N=8: v_A = \frac{0.954(1^2) - 0.989(2^2)}{1^2 - 2^2} = 1.001$$

(d) QM6 results in Fig. 6.6-1: $O(h^2)$ for disp., $O(h)$ for stress

$$N=2, N=4: v_c = \frac{0.884(1^2) - 0.967(2^2)}{1^2 - 2^2} = 0.995$$

$$\sigma_A = \frac{0.840(1) - 0.978(2)}{1 - 2} = 1.116$$

$$\sigma_B = \frac{0.788(1) - 0.926(2)}{1 - 2} = 1.064$$

But if we assume $O(h^3)$ for disp. and $O(h^2)$ for stress, then

$$N=2, N=4: v_c = \frac{0.884(1^3) - 0.967(2^3)}{1^3 - 2^3} = 0.979$$

$$\sigma_A = \frac{0.840(1^2) - 0.978(2^2)}{1^2 - 2^2} = 1.024$$

$$\sigma_B = \frac{0.788(1^2) - 0.962(2^2)}{1^2 - 2^2} = 1.020$$

(continues)

9.7-5 (concluded)

(e) Q4 results in Fig. 6.6-1: $O(h^2)$ for disp., $O(h)$ for stress

$$N=2, N=4: \quad v_c = \frac{0.498(1^2) - 0.769(2^2)}{1^2 - 2^2} = 0.859$$

$$\sigma_A = \frac{0.558(1) - 0.830(2)}{1 - 2} = 1.102$$

$$\sigma_B = \frac{0.457(1) - 0.753(2)}{1 - 2} = 1.049$$

9.7-6

Displacement: error is $O(h^3)$. Apply Eq. 9.7-1:

$$\frac{0.0035(1^3) - 0.0041(2^3)}{1^3 - 2^3} = 0.00419$$

Estimated error of mesh 2:

$$\frac{0.00410 - 0.00419}{0.00419} 100\% = -2.05\%$$

Stress: error is $O(h^2)$. Apply Eq. 9.7-1:

$$\frac{74.23(1^2) - 89.03(2^2)}{1^2 - 2^2} = 93.96$$

Estimated error of mesh 2:

$$\frac{89.03 - 93.96}{93.96} 100\% = -5.25\%$$

9.7-7

Refinement is not regular. Count the number of elements:

$N = 9$ in the coarse mesh

$N = 62$ in the finer mesh

Number of elements per side $\approx \sqrt{N}$. Thus

$N_{es} \approx 3$ coarse mesh

$N_{es} \approx 7.874$ finer mesh

Stress error is $O(h)$

Element side lengths $\approx$ proportional to $\frac{1}{N_{es}}$. Apply Eq. 9.7-1:

$$\phi_{\infty} = \frac{\phi_1 h_2^q - \phi_2 h_1^q}{h_2^q - h_1^q} = \frac{\phi_1 - \phi_2 (h_1/h_2)^q}{1 - (h_1/h_2)^q}$$

$$\sigma_E = \frac{2.68 - 3.16 (7.874/3)^2}{1 - (7.874/3)^2} = 3.24$$

(if stress error is $O(h^2)$)

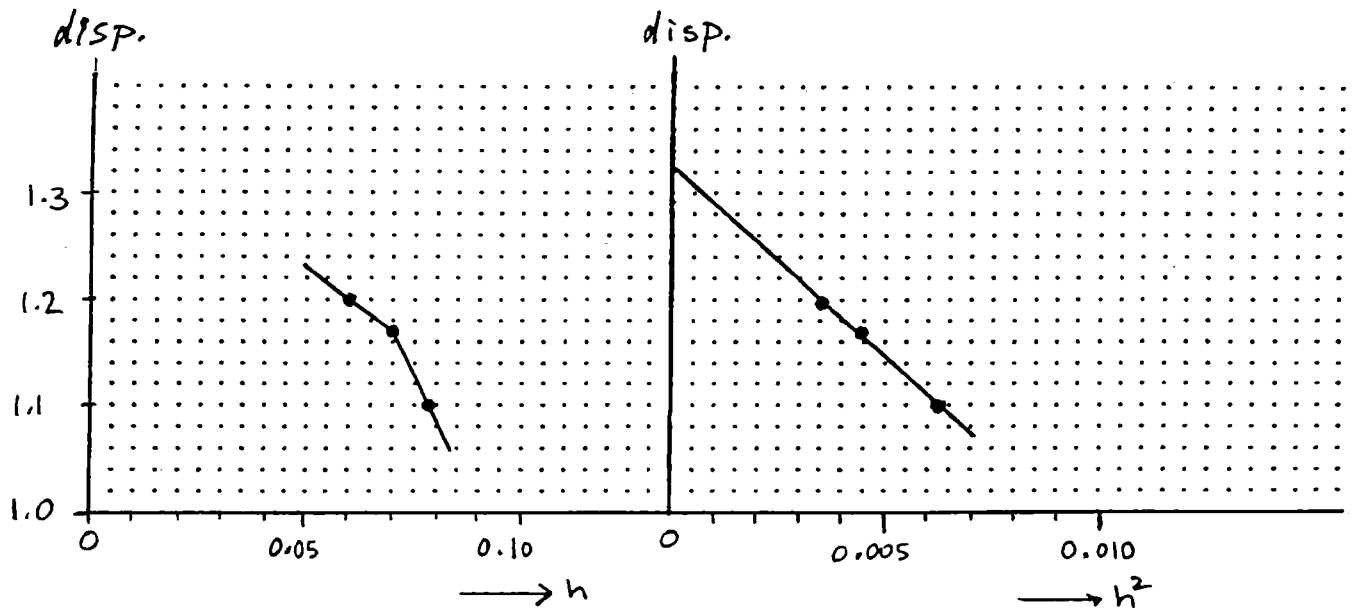
Est. error of finer mesh: $\frac{3.16 - 3.24}{3.24} 100\% = -2.5\%$

9.7-8

(a) No-regular refinement requires $h_{i+1} = h_i/m$, where m is an integer.

(b) With $h = \frac{1}{N^{1/3}}$,

mesh	no. of d.o.f., N	disp.	h	h^2
1	2014	1.10	0.0792	0.00627
2	3342	1.17	0.0669	0.00447
3	4560	1.20	0.0603	0.00364



Convergence appears to be $O(h^2)$. This is reasonable: eight-node bricks contain a complete linear polynomial for displacement, so quadratic convergence is expected.

(c) Extrapolated displacement:

$$\phi = \frac{0.00364(1.17) - 0.00447(1.20)}{0.00364 - 0.00447} = 1.332$$

Est. error of finest mesh:

$$e = \frac{1.20 - 1.332}{1.332} 100\% = -9.9\%$$

9.9-1 (a)

$$F_G = \sum \int (\sigma^* - \sigma)^2 dV \quad \text{with } \sigma^* = \underline{\underline{N}} \underline{\underline{\sigma}}_n^* \text{ yields}$$

$$F_G = \sum \int (\underline{\underline{N}} \underline{\underline{\sigma}}_n^* - \sigma)^T (\underline{\underline{N}} \underline{\underline{\sigma}}_n^* - \sigma) dV = \sum \int (\underline{\underline{\sigma}}_n^{*T} \underline{\underline{N}}^T - \sigma) (\underline{\underline{N}} \underline{\underline{\sigma}}_n^* - \sigma) dV$$

$$F_G = \sum \int (\underline{\underline{\sigma}}_n^{*T} \underline{\underline{N}}^T \underline{\underline{N}} \underline{\underline{\sigma}}_n^* - \underline{\underline{\sigma}}_n^{*T} \underline{\underline{N}}^T \sigma - \sigma \underline{\underline{N}} \underline{\underline{\sigma}}_n^* + \sigma^2) dV$$

$$F_G = \sum \int (\underline{\underline{\sigma}}_n^{*T} \underline{\underline{N}}^T \underline{\underline{N}} \underline{\underline{\sigma}}_n^* - 2 \underline{\underline{\sigma}}_n^{*T} \underline{\underline{N}}^T \sigma + \sigma^2) dV$$

$$\frac{\partial F_G}{\partial \underline{\underline{\sigma}}_n^*} = 0 = \sum \int (2 \underline{\underline{N}}^T \underline{\underline{N}} \underline{\underline{\sigma}}_n^* - 2 \underline{\underline{N}}^T \sigma) dV$$

$$\sum \int \underline{\underline{N}}^T \underline{\underline{N}} dV \underline{\underline{\sigma}}_n^* = \sum \int \underline{\underline{N}}^T \sigma dV$$

$$\left(\sum \int \underline{\underline{N}}^T \underline{\underline{N}} dV \right) \{ \underline{\underline{\sigma}}_n^* \}_G = \sum \int \underline{\underline{N}}^T \sigma dV$$

$$(b) F_P = \sum (\sigma^* - \sigma)^2 \quad \text{with } \sigma^* = \underline{\underline{P}} \underline{\underline{a}} \text{ yields}$$

$$F_P = \sum (\underline{\underline{P}} \underline{\underline{a}} - \sigma)^T (\underline{\underline{P}} \underline{\underline{a}} - \sigma) = \sum (\underline{\underline{a}}^T \underline{\underline{P}}^T - \sigma) (\underline{\underline{P}} \underline{\underline{a}} - \sigma)$$

$$F_P = \sum (\underline{\underline{a}}^T \underline{\underline{P}}^T \underline{\underline{P}} \underline{\underline{a}} - \underline{\underline{a}}^T \underline{\underline{P}}^T \sigma - \sigma \underline{\underline{P}} \underline{\underline{a}} + \sigma^2)$$

$$F_P = \sum (\underline{\underline{a}}^T \underline{\underline{P}}^T \underline{\underline{P}} \underline{\underline{a}} - 2 \underline{\underline{a}}^T \underline{\underline{P}}^T \sigma + \sigma^2)$$

$$\frac{\partial F_P}{\partial \underline{\underline{a}}} = 0 = \sum (2 \underline{\underline{P}}^T \underline{\underline{P}} \underline{\underline{a}} - 2 \underline{\underline{P}}^T \sigma)$$

$$\sum [\underline{\underline{P}}^T \underline{\underline{P}}] \underline{\underline{a}} = \sum \underline{\underline{P}}^T \sigma$$

9.9-2

$$(a) [A] = [P]_A^T [P]_A + [P]_B^T [P]_B \quad \text{with} \quad \sigma = [P] \{a\} = \begin{bmatrix} 1 & x \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$$

$$[A] = \begin{Bmatrix} 1 \\ L/2 \end{Bmatrix} \begin{bmatrix} 1 & L/2 \end{bmatrix} + \begin{Bmatrix} 1 \\ 3L/2 \end{Bmatrix} \begin{bmatrix} 1 & 3L/2 \end{bmatrix}$$

$$[A] = \begin{bmatrix} 1 & L/2 \\ L/2 & L^2/4 \end{bmatrix} + \begin{bmatrix} 1 & 3L/2 \\ 3L/2 & 9L^2/4 \end{bmatrix} = \begin{bmatrix} 2 & 2L \\ 2L & 2.5L^2 \end{bmatrix}$$

$$\{b\} = [P]_A^T \sigma_{xA} + [P]_B^T \sigma_{xB} = \begin{Bmatrix} 1 \\ L/2 \end{Bmatrix} (1) + \begin{Bmatrix} 1 \\ 3L/2 \end{Bmatrix} (3) = \begin{Bmatrix} 4 \\ 7L/2 \end{Bmatrix}$$

$$(b) \begin{bmatrix} 2 & 2L \\ 2L & 2.5L^2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 7L/2 \end{Bmatrix}$$

$$\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \frac{1}{L^2} \begin{bmatrix} 2.5L^2 & -2L \\ -2L & 2 \end{bmatrix} \begin{Bmatrix} 4 \\ 7L/2 \end{Bmatrix} = \frac{1}{L^2} \begin{Bmatrix} 3L^2 \\ -L \end{Bmatrix} = \begin{Bmatrix} 3 \\ -1/L \end{Bmatrix}$$

$$\text{At } x = L/2, \quad \sigma = \begin{bmatrix} 1 & L/2 \end{bmatrix} \begin{Bmatrix} 3 \\ -1/L \end{Bmatrix} = 2 \quad \checkmark$$

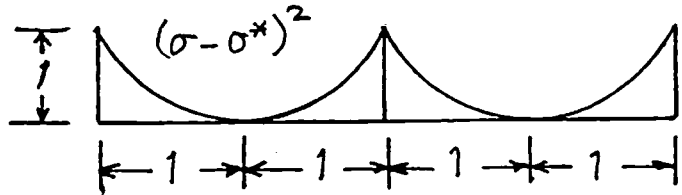
9.10-1

For convenience, assume $E = 1$.

$$U^2 = 2^2(2) + 4^2(2) = 40$$

The difference between average stress and element stress is represented by four triangles, each of span 1 and altitude (stress) 1. The plot of stress difference squared therefore consists of four parabolas:

Area under each is $\frac{1}{3}$



$$\int (\sigma - \sigma^*)^2 dx = 4\left(\frac{1}{3}\right) = 1.333 = e^2$$

$$U^2 + e^2 = 40 + 1.333 = 41.33$$

From the average stress plot,

$$(U^*)^2 = \int_0^4 (1+x)^2 dx = \left[\frac{1}{3}(1+x)^3 \right]_0^4 = \frac{125}{3} = 41.67$$

close!

$$\eta = \left[\frac{1.333}{41.33} \right]^{1/2} = 0.18$$

9.10-2

$$u = 2x - 0.1x^3$$

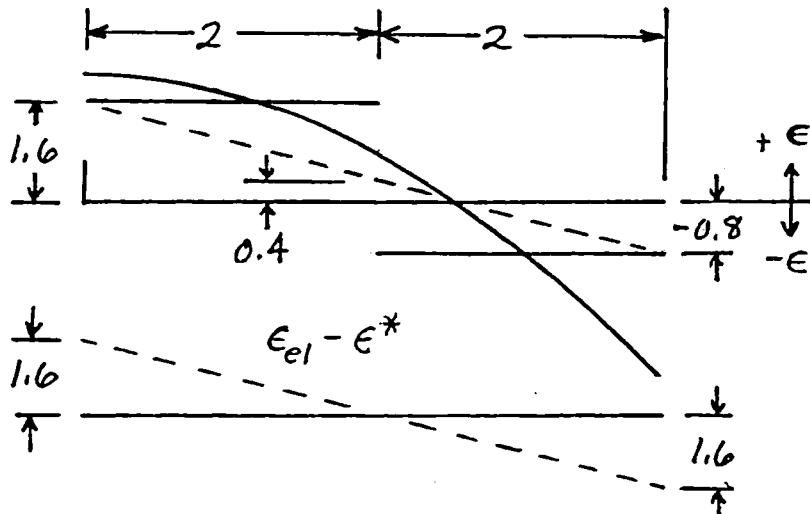
(a)

$$\epsilon_x = 2 - 0.3x^2$$

x	u	ϵ_{el}	$\epsilon^* \text{ (nodal ave.)}$
0	0	1.6	1.6
2	3.2	-0.8	0.4
4	1.6	-0.8	-0.8

A and E do not matter because the bar is uniform.

$$\epsilon_{el} = \frac{u_{i+1} - u_i}{L_i} = \frac{u_{i+1} - u_i}{2}$$



$$\|U\|^2 = 2(1.6)^2 + 2(-0.8)^2 = 6.4 \quad \|U\| = 2.530$$

$$\|e\|^2 = \frac{1}{3} 2(1.2)^2 + \frac{1}{3} 2(1.2)^2 = 1.92 \quad \|e\| = 1.386$$

$$\begin{aligned} \text{Exact } U: \int_0^4 \epsilon_x^2 dx &= \int_0^4 (4 - 1.2x^2 + 0.09x^4) dx \\ &= \left(4x - 0.4x^3 + \frac{0.09}{5}x^5 \right)_0^4 = 8.832 \end{aligned}$$

which compares with $\|U\|^2 + \|e\|^2 = 6.4 + 1.92 = 8.32$

$$\eta = \left[\frac{1.92}{8.832} \right]^{1/2} = 0.480$$

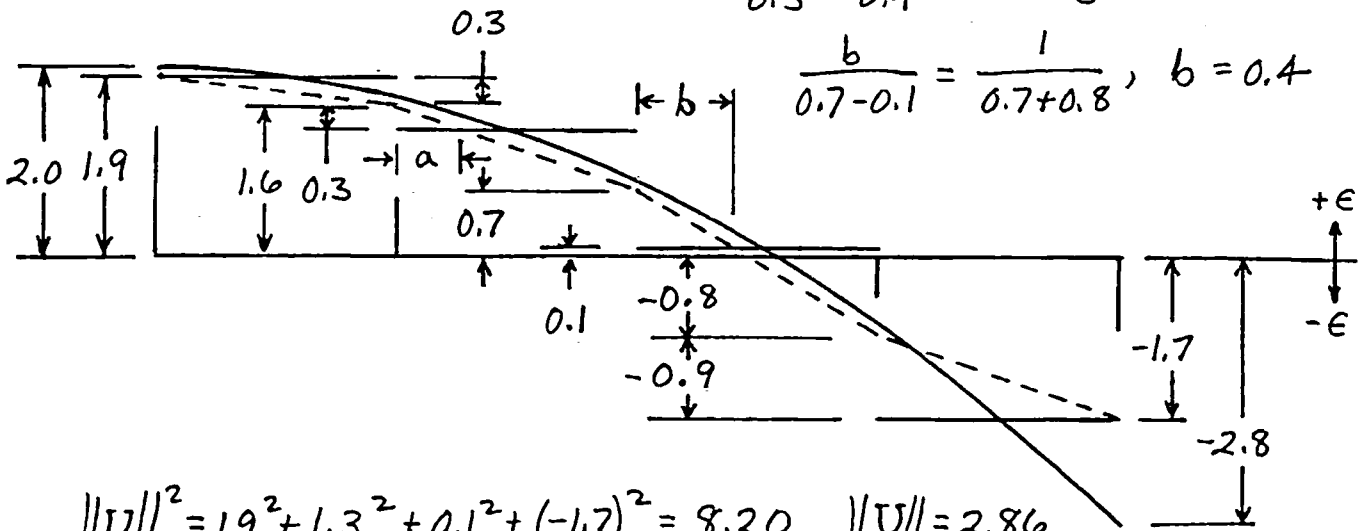
	x	u	ϵ_{el}	$\epsilon^* \text{ (nodal ave.)}$
(b) Four elements, $L=1$ for each.	0	0	1.9	1.9
	1	1.9	1.3	1.6
	2	3.2	0.1	0.7
	3	3.3	-1.7	-0.8
	4	1.6	-1.7	-1.7
	(continues)			

9.10-2 (concluded)

$L = 1$ for each element

$$\frac{a}{0.3} = \frac{1}{0.9}, \quad a = \frac{1}{3}$$

$$\frac{b}{0.7-0.1} = \frac{1}{0.7+0.8}, \quad b = 0.4$$



$$\|U\|^2 = 1.9^2 + 1.3^2 + 0.1^2 + (-1.7)^2 = 8.20, \quad \|U\| = 2.86$$

For $\|e\|^2$, we square a number of triangles, obtaining parabolas, whose area is (base)(height)/3, where here (base) = 1, a, 1-a, b, 1-b, 1.

$$\|e\|^2 = \frac{1}{3} \left[0.3^2 + 0.3^2 \left(\frac{1}{3} \right) + 0.6^2 \left(\frac{2}{3} \right) + 0.6^2 (0.4) + 0.9^2 (0.6) + 0.9^2 \right] = 0.60$$

$$\|e\| = 0.775$$

$$\|U\|^2 + \|e\|^2 = 8.20 + 0.60 = 8.80$$

(close to exact $U = 8.832$ on previous page)

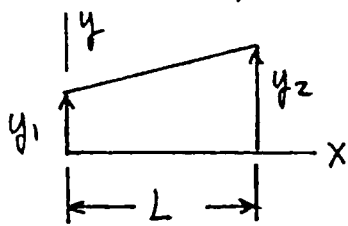
$$\eta = \left[\frac{0.60}{8.80} \right]^{1/2} = 0.261$$

(c) $\|e\|$ is roughly halved, so convergence rate of strains and stresses appears to be about $O(h)$, as should be expected.

However the first mesh is so coarse that the true rate may not yet have appeared.

9.10-3

For a trapezoid,



$$\int_0^L y^2 dx = \frac{L}{3} (y_1^2 + y_1 y_2 + y_2^2)$$

For a triangle, say $y_2 = 0$, $\int_0^L y^2 dx = \frac{L}{3} y_1^2$

For the coarse mesh, Prob. 9.10-2a,

$$\|U^*\|^2 = \frac{2}{3} [1.6^2 + 1.6(0.4) + 0.4^2] + \frac{2/3}{3} 0.4^2 + \frac{4/3}{3} (-0.8)^2 = 2.56$$

$$\|U^*\| = 1.60$$

For the finer mesh, Prob. 9.10-2b,

$$\begin{aligned} \|U^*\|^2 &= \frac{1}{3} [1.9^2 + 1.9(1.6) + 1.6^2] + \frac{1}{3} [1.6^2 + 1.6(0.7) + 0.7^2] \\ &\quad + \frac{0.4}{3} 0.7^2 + \frac{0.6}{3} (-0.8)^2 + \frac{1}{3} [(-0.8)^2 + (-0.8)(-1.7) + (-1.7)^2] \\ &= 6.283 \end{aligned}$$

$$\|U^*\| = 2.51$$

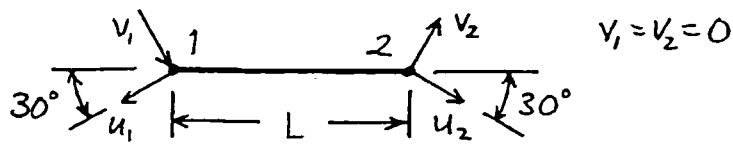
Values of $\|U\|^2 + \|e\|^2$, from Prob. 9.10-2:

part (a), 8.32

part (b), 8.80

These values do not agree well with foregoing values of $\|U^*\|^2$ because smoothing by linear interpolation from nodal average values is not very accurate.

10.9-1



Let F = axial force in bar. $F = k(0.866u_1)$
 where $k = AE/L$. Component of F in
 u_1 direction is $0.866F = 0.75ku_1$.

Hence $[K]\{D\} = \{R\}$ is

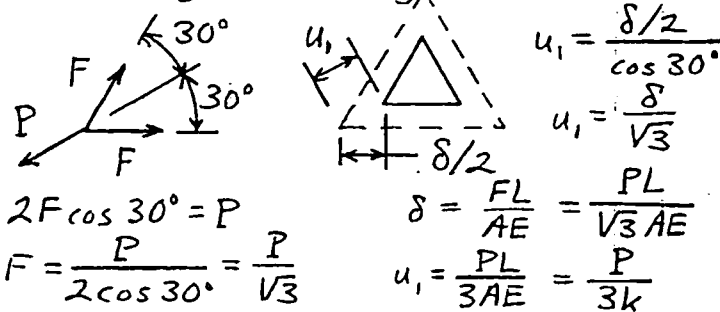
$$0.75k \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$

where P appears only
 once so as not to double the load.

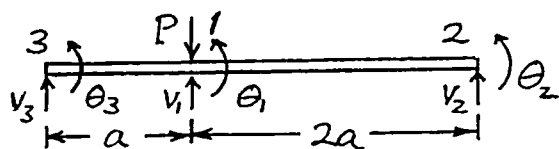
Also $\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_1 = [T]u_1$

$$\begin{aligned} [T]^T [K] [T] &= 3k \\ [T]^T \{R\} &= P \end{aligned} \quad \left. \vphantom{\begin{aligned} [T]^T [K] [T] &= 3k \\ [T]^T \{R\} &= P \end{aligned}} \right\} 3ku_1 = P, \quad u_1 = \frac{P}{3k}$$

Check by elementary methods:



10.9-2



For left portion, $[k]$ is (note signs)

$$\frac{EI}{a^3} \begin{bmatrix} 12 & -6a & -12 & -6a \\ -6a & 4a^2 & 6a & 2a^2 \\ -12 & 6a & 12 & 6a \\ -6a & 2a^2 & 6a & 4a^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_3 \\ \theta_3 \end{Bmatrix}$$

For right portion, $[k]$ is

$$\frac{EI}{(2a)^3} \begin{bmatrix} 12 & 6(2a) & -12 & 6(2a) \\ 6(2a) & 4(2a)^2 & -6(2a) & 2(2a)^2 \\ -12 & -6(2a) & 12 & -6(2a) \\ 6(2a) & 2(2a)^2 & -6(2a) & 4(2a)^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

Set $v_2 = v_3 = 0$ (supported); combine $[k]$'s

$$\frac{EI}{8a^3} \begin{bmatrix} 96 & -48a & 0 & -48a \\ -48a & 32a^2 & 0 & 16a^2 \\ 0 & 0 & 0 & 0 \\ -48a & 16a^2 & 0 & 32a^2 \end{bmatrix} + \frac{EI}{8a^3} \begin{bmatrix} 12 & 12a & 12a & 0 \\ 12a & 16a^2 & 8a^2 & 0 \\ 12a & 8a^2 & 16a^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

$$[K] = \frac{EI}{8a^3} \begin{bmatrix} 108 & -36a & 12a^2 & -48a \\ -36a & 48a^2 & 8a^2 & 16a^2 \\ 12a & 8a^2 & 16a^2 & 0 \\ -48a & 16a^2 & 0 & 32a^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix}$$

$$\begin{Bmatrix} v_1 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ \theta_2 \end{Bmatrix} = [T] \begin{Bmatrix} v_1 \\ \theta_1 \\ \theta_2 \end{Bmatrix}$$

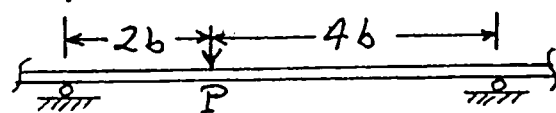
$[T]^T [K] [T]$ yields

$$\frac{EI}{8a^3} \begin{bmatrix} 108 & -36a & -36a \\ -36a & 48a^2 & 24a^2 \\ -36a & 24a^2 & 48a^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} -P \\ 0 \\ 0 \end{Bmatrix}$$

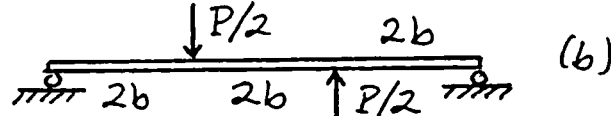
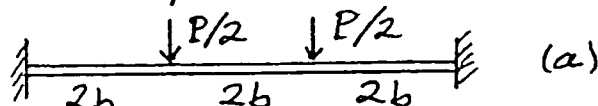
Last 2 eqs. yield $\theta_1 = \theta_2$. Hence the

first 2 eqs. yield $108v_1 - 72a\theta_1 = \frac{8Pa^3}{EI}$
 $-36v_1 + 72a\theta_1 = 0$
 $72v_1 = -\frac{8Pa^3}{EI}$
 $v_1 = -\frac{Pa^3}{9EI}$ $\downarrow$, $\theta_1 = \theta_2 = \frac{v_1}{2a} = -\frac{Pa^2}{18EI}$

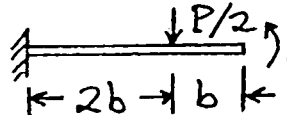
Check by elementary methods: let $a = 2b$



is equal to sum of -



(a) is same as



Now apply beam formulas:

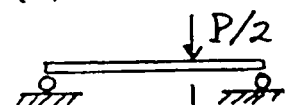
$$\theta_c = 0 = \frac{(P/2)(2b)^2}{2EI} - \frac{M_c(3b)}{EI}$$

$$\therefore M_c = Pb/3 \quad \text{At load } P/2,$$

$$v_a = \frac{(P/2)(2b)^3}{3EI} - \frac{M_c(2b)^2}{2EI}$$

$$v_a = \frac{4Pb^3}{3EI} - \frac{2Pb^3}{3EI} = \frac{2Pb^3}{3EI}$$

(b) is same as



At load $P/2$,

$$v_b = \frac{(P/2)(2b)^2 b^2}{3EI(3b)} = \frac{2Pb^3}{9EI}$$

Net result:

$$v_a + v_b = \frac{Pb^3}{EI} \left(\frac{2}{3} + \frac{2}{9} \right) = \frac{8Pb^3}{9EI} = \frac{Pa^3}{9EI}$$

(downward)

11.2-1

$$d(KE) = \frac{1}{2} v^2 dm, \quad KE = \int \frac{1}{2} v^2 dm$$

$$\text{disp.} = \underline{\underline{N}} \underline{\underline{d}}, \quad v = \underline{\underline{N}} \underline{\underline{\dot{d}}}, \quad dm = \rho dV$$

$$v^2 = v^T v = \underline{\underline{\dot{d}}}^T \underline{\underline{N}}^T \underline{\underline{N}} \underline{\underline{\dot{d}}}$$

$$KE = \frac{1}{2} \underline{\underline{\dot{d}}}^T \underbrace{\int \underline{\underline{N}}^T \underline{\underline{N}} \rho dV}_{\underline{\underline{m}}} \underline{\underline{\dot{d}}} = \frac{1}{2} \underline{\underline{\dot{d}}}^T \underline{\underline{m}} \underline{\underline{\dot{d}}}$$

more generally, ^m with displacement components u, v , and w ,

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \underline{\underline{N}} \underline{\underline{d}}, \quad \text{velocities are } \begin{Bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{Bmatrix} = \underline{\underline{N}} \underline{\underline{\dot{d}}}$$

The resultant velocity squared is

$$\begin{Bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{Bmatrix}^T \begin{Bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{Bmatrix} = \underline{\underline{\dot{d}}}^T \underline{\underline{N}}^T \underline{\underline{N}} \underline{\underline{\dot{d}}} \quad (\text{as above})$$

11.2-2

$$\{\underline{u}\} = [\underline{N}]\{\underline{d}\}, \text{ so } \{\underline{\ddot{u}}\} = [\underline{N}]\{\underline{\ddot{d}}\}.$$

$$\{\underline{r}\} = [\underline{m}]\{\underline{\ddot{d}}\} = \int [\underline{N}]^T [\underline{N}] \rho \{\underline{\ddot{d}}\} dV = \int [\underline{N}]^T \rho \{\underline{\ddot{u}}\} dV$$

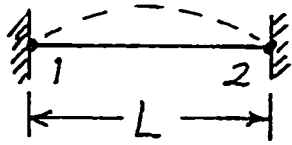
But $\rho\{\underline{\ddot{u}}\}$ are "effective" body forces, so

$$\{\underline{r}\} = \int [\underline{N}]^T \{\underline{F}\} dV \text{ as in Chapters 3 \& 4.}$$

11.3-1

(a) No. $m_{ii} = \int \rho N_i^2 dV$, and since ρ , N_i^2 , and dV are all positive, $m_{ii} > 0$.

(b) The half-wave of a typical mode is spanned by one element:



Nodes have only rotational d.o.f.

$$\left(\frac{EI}{L^3} \begin{bmatrix} 4L^2 & 2L^2 \\ 2L^2 & 4L^2 \end{bmatrix} - \omega^2 \frac{mL^2}{24} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{Bmatrix} \theta_{21} \\ \theta_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$\theta_{21} = -\theta_{22}$, so 1st eq. becomes

$$\left(\frac{2EI}{L} - \omega^2 \frac{mL^2}{24} \right) \theta_{21} = 0$$

$$\omega^2 = \frac{48EI}{mL^3}, \quad \omega = 6.93 \sqrt{\frac{EI}{mL^3}}$$

11.3-2

For the three respective motions,
correct kinetic energies are

$$KE_1 = \frac{1}{2} m v^2, \quad KE_2 = \frac{1}{2} \frac{m L^2}{12} \Omega^2, \quad KE_3 = \frac{1}{2} \frac{m L^2}{3} \Omega^2$$

where $m = \rho A L$. In terms of nodal d.o.f.
and mass matrix $\underline{m}$, $KE = \frac{1}{2} \dot{\underline{d}}^T \underline{m} \dot{\underline{d}}$.

$$(a) \quad \underline{m} = \frac{m}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\dot{\underline{d}}_1 = v \begin{bmatrix} 1 & 1 \end{bmatrix}^T, \quad KE_1 = \frac{m v^2}{2} \quad \checkmark$$

$$\dot{\underline{d}}_2 = \frac{\Omega L}{2} \begin{bmatrix} -1 & 1 \end{bmatrix}^T, \quad KE_2 = \frac{m L^2 \Omega^2}{8} \quad \times$$

$$\dot{\underline{d}}_3 = \Omega L \begin{bmatrix} 0 & 1 \end{bmatrix}^T, \quad KE_3 = \frac{m L^2 \Omega^2}{2} \quad \times$$

$$(b) \quad \underline{m} = \frac{m}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\dot{\underline{d}}_1 = v \begin{bmatrix} 1 & 1 \end{bmatrix}^T, \quad KE_1 = \frac{m v^2}{2} \quad \checkmark$$

$$\dot{\underline{d}}_2 = \frac{\Omega L}{2} \begin{bmatrix} -1 & 1 \end{bmatrix}^T, \quad KE_2 = \frac{m L^2 \Omega^2}{24} \quad \checkmark$$

$$\dot{\underline{d}}_3 = \Omega \begin{bmatrix} 0 & 1 \end{bmatrix}^T, \quad KE_3 = \frac{m L^2 \Omega^2}{6} \quad \checkmark$$

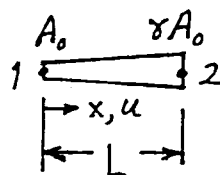
$$(c) \quad \underline{m} = \frac{m}{420} \left[\text{as in Eq. 11.3-5} \right]$$

$$\dot{\underline{d}}_1 = v \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}^T, \quad KE_1 = \frac{m v^2}{2} \quad \checkmark$$

$$\dot{\underline{d}}_2 = \frac{\Omega L}{2} \begin{bmatrix} -1 & 2 & \frac{L}{2} & 1 \end{bmatrix}^T, \quad KE_2 = \frac{m L^2 \Omega^2}{24} \quad \checkmark$$

$$\dot{\underline{d}}_3 = \Omega \begin{bmatrix} 0 & 1 & L & 1 \end{bmatrix}^T, \quad KE_3 = \frac{m L^2 \Omega^2}{6} \quad \checkmark$$

11.3-3


 $u = [N] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}, A = [N] \begin{Bmatrix} A_0 \\ \gamma A_0 \end{Bmatrix}$
 where $[N] = \begin{bmatrix} \frac{L-x}{L} & \frac{x}{L} \end{bmatrix}$

$$[m] = \int \rho [N]^T [N] A dx, A = \frac{A_0}{L} [L + (\gamma - 1)x]$$

$$[m] = \frac{\rho A_0}{L^3} \int_0^L \begin{bmatrix} (L-x)^2 & x(L-x) \\ x(L-x) & x^2 \end{bmatrix} [L + (\gamma - 1)x] dx$$

$$\int_0^L (L-x)^2 dx = \int_0^L x^2 dx = \frac{L^3}{3}, \int_0^L x(L-x) dx = \frac{L^3}{6}$$

$$\int_0^L (L-x)^2 x dx = \frac{L^4}{12}, \int_0^L x^2(L-x) dx = \frac{L^4}{12}, \int_0^L x^3 dx = \frac{L^4}{4}$$

$$[m] = \rho A_0 L \left(\begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} + (\gamma - 1) \begin{bmatrix} 1/12 & 1/12 \\ 1/12 & 1/4 \end{bmatrix} \right)$$

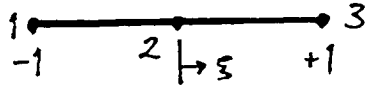
$$[m] = \frac{\rho A_0 L}{12} \begin{bmatrix} 4 + (\gamma - 1) & 2 + (\gamma - 1) \\ 2 + (\gamma - 1) & 4 + 3(\gamma - 1) \end{bmatrix}$$

(c) $\{\ddot{u}\} = \begin{Bmatrix} a \\ 0 \\ a \\ 0 \end{Bmatrix}, [m]\{\ddot{u}\} = \frac{ma}{6} \begin{Bmatrix} 3 \\ 0 \\ 3 \\ 0 \end{Bmatrix} = \{r\}$

$$\sum F_x = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \{r\} = \frac{ma}{6} 6 = ma \checkmark$$

$$[m]\{\ddot{u}\} = \frac{ma}{2} \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{Bmatrix} = \{r\}, \sum F_x = \frac{ma}{2} 2 = ma \checkmark$$

11.3-4



$$[\tilde{m}] = \int_{-1}^1 [N]^T [N] \rho A \frac{L}{2} d\xi, [N] = \left[\frac{-\xi + \xi^2}{2}, 1 - \xi^2, \frac{\xi + \xi^2}{2} \right]$$

$$\int_{-1}^1 \left(\frac{-\xi + \xi^2}{2} \right)^2 d\xi = \frac{8}{30}, \int_{-1}^1 \frac{-\xi + \xi^2}{2} (1 - \xi^2) d\xi = \frac{4}{30}$$

$$\int_{-1}^1 \frac{-\xi + \xi^2}{2} \left(\frac{\xi + \xi^2}{2} \right) d\xi = -\frac{2}{30}, \int_{-1}^1 (1 - \xi^2)^2 d\xi = \frac{32}{30}$$

$$\int_{-1}^1 (1 - \xi^2) \frac{\xi + \xi^2}{2} d\xi = \frac{4}{30}, \int_{-1}^1 \left(\frac{\xi + \xi^2}{2} \right)^2 d\xi = \frac{8}{30}$$

With $m = \rho AL$,

$$[\tilde{m}] = \frac{m}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

11.3-5

$$\text{Exact: } \frac{1}{2} I \Omega^2 = \frac{1}{2} \frac{m L^2}{12} \Omega^2 = \frac{1}{24} m L^2 \Omega^2 = 0.04167 m L^2 \Omega^2$$

$$\text{FEA: } KE = \frac{1}{2} \{\dot{\underline{d}}\}^T [\underline{m}] \{\dot{\underline{d}}\}$$

$$\text{Here } \{\dot{\underline{d}}\} = \begin{Bmatrix} -\Omega L/2 \\ \Omega \\ \Omega L/2 \\ \Omega \end{Bmatrix}$$

$$(a) \text{ HRZ: } [\underline{m}] \{\dot{\underline{d}}\} = \frac{m}{78} \begin{bmatrix} 39 & & & \\ & L^2 & & \\ & & 39 & \\ & & & L^2 \end{bmatrix} \{\dot{\underline{d}}\} = \frac{m \Omega}{78} \begin{Bmatrix} -39L/2 \\ L^2 \\ 39L/2 \\ L^2 \end{Bmatrix}$$

$$KE = \frac{1}{2} \{\dot{\underline{d}}\}^T ([\underline{m}] \{\dot{\underline{d}}\}) = \frac{1}{2} \frac{m \Omega^2 L^2}{78} (21.5) = 0.1378 m \Omega^2 L^2$$

Error is 231% (high)

(b) Particle-lumped:

$$[\underline{m}] \{\dot{\underline{d}}\} = \frac{m}{2} \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 1 & \\ & & & 0 \end{bmatrix} \{\dot{\underline{d}}\} = \frac{m \Omega}{2} \begin{Bmatrix} -L/2 \\ 0 \\ L/2 \\ 0 \end{Bmatrix}$$

$$KE = \frac{1}{2} \{\dot{\underline{d}}\}^T ([\underline{m}] \{\dot{\underline{d}}\}) = \frac{1}{2} m \Omega^2 L^2 \left(\frac{1}{4}\right) = 0.1250 m \Omega^2 L^2$$

Error is 200% (high)

11.3-6

By FEA:

$$KE = \frac{1}{2} \{\dot{\alpha}\}^T [\underline{m}] \{\dot{\alpha}\} = \frac{1}{2} \begin{bmatrix} -\frac{\Omega L}{2} & \Omega & \frac{\Omega L}{2} & \Omega \end{bmatrix} \begin{bmatrix} m/2 & & & \\ & m\alpha L^2 & & \\ & & m/2 & \\ & & & m\alpha L^2 \end{bmatrix} \begin{Bmatrix} -\Omega L/2 \\ \Omega \\ \Omega L/2 \\ \Omega \end{Bmatrix}$$

$$KE = \frac{m\Omega^2}{2} \begin{bmatrix} -\frac{L}{2} & 1 & \frac{L}{2} & 1 \end{bmatrix} \begin{Bmatrix} -L/4 \\ \alpha L^2 \\ L/4 \\ \alpha L^2 \end{Bmatrix} = \frac{m\Omega^2 L^2}{2} \left(\frac{1}{4} + 2\alpha \right)$$

$$\text{Exact KE: } \frac{1}{2} I \Omega^2 = \frac{1}{2} \frac{mL^2}{12} \Omega^2$$

$$\text{Equate KE's: } \frac{1}{12} = \frac{1}{4} + 2\alpha \quad \text{so } \alpha = -\frac{1}{12}$$

For a one-element s.s. beam, only the (negative) rotational masses remain. Hence ω^2 is negative and natural frequencies ω are imaginary. We conclude that negative masses are dangerous.

11.3-7

(a) Consider x-direction motion first.

$$\{\text{nodal forces}\} = [\underline{m}_x] [\ddot{u}_1, \ddot{u}_2, \ddot{u}_3]^T$$

$$u = [\underline{N}] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \text{ where } [\underline{N}] = [\xi_1, \xi_2, \xi_3] \text{ and}$$

ξ_1, ξ_2, ξ_3 are area coordinates (Chap. 5).

$$[\underline{m}_x] = \rho t \int_A [\underline{N}]^T [\underline{N}] dA = \rho t \int_A \begin{bmatrix} \xi_1^2 & \xi_1 \xi_2 & \xi_1 \xi_3 \\ \xi_1 \xi_2 & \xi_2^2 & \xi_2 \xi_3 \\ \xi_1 \xi_3 & \xi_2 \xi_3 & \xi_3^2 \end{bmatrix} dA$$

Use Eq. to
integrate; result is $[\underline{m}_x] = \frac{\rho t A}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Since the y-dir. field has the same form,
the $[\underline{m}]$ that operates on $[u, v, u_2, v_2, u_3, v_3]^T$ is

(b)

$$[\underline{m}] = \frac{\rho t A}{12} \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix}$$

11.3-8 For the entire element, $J = \frac{A}{4} = \frac{ab}{4}$

$$m_{ij} = \iint \rho N_i N_j t \, dx \, dy = \rho t J \int_{-1}^1 \int_{-1}^1 N_i N_j \, d\xi \, d\eta$$

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta), \quad N_2 = \frac{1}{4}(1+\xi)(1-\eta), \quad N_3 = \frac{1}{4}(1+\xi)(1+\eta), \quad N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

$$N_1^2 = \frac{1}{4^2}(1-\xi)^2(1-\eta)^2 = \frac{1}{4^2}(1-2\xi+\xi^2)(1-2\eta+\eta^2)$$

$$N_1 N_2 = \frac{1}{4^2}(1-\xi^2)(1-\eta)^2 = \frac{1}{4^2}(1-\xi^2)(1-2\eta+\eta^2)$$

$$N_1 N_3 = \frac{1}{4^2}(1-\xi^2)(1-\eta^2)$$

$$N_1 N_4 = \frac{1}{4^2}(1-\xi)^2(1-\eta^2) = \frac{1}{4^2}(1-2\xi+\xi^2)(1-\eta^2)$$

$$N_2 N_4 = \frac{1}{4^2}(1-\xi^2)(1-\eta^2)$$

$$N_2 N_3 = \frac{1}{4^2}(1+\xi)^2(1-\eta^2) = \frac{1}{4^2}(1+2\xi+\xi^2)(1-\eta^2)$$

$$N_2^2 = \frac{1}{4^2}(1+\xi)^2(1-\eta)^2 = \frac{1}{4^2}(1+2\xi+\xi^2)(1-2\eta+\eta^2)$$

$$N_3^2 = \frac{1}{4^2}(1+\xi)^2(1+\eta)^2 = \frac{1}{4^2}(1+2\xi+\xi^2)(1+2\eta+\eta^2)$$

$$N_3 N_4 = \frac{1}{4^2}(1-\xi^2)(1+\eta)^2 = \frac{1}{4^2}(1-\xi^2)(1+2\eta+\eta^2)$$

$$N_4^2 = \frac{1}{4^2}(1-\xi)^2(1+\eta)^2 = \frac{1}{4^2}(1-2\xi+\xi^2)(1+2\eta+\eta^2)$$

Note that first powers of ξ and η integrate to zero. Therefore

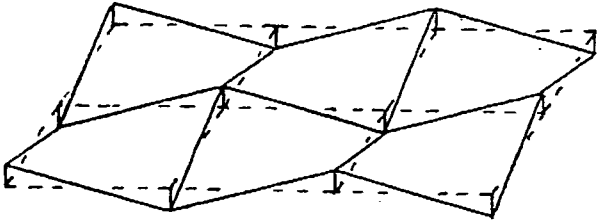
$$m_{11} = m_{22} = m_{33} = m_{44}, \quad m_{12} = m_{14} = m_{23} = m_{34}, \quad m_{13} = m_{24}$$

$$m_{11} = \frac{\rho abt}{4^3} \int_{-1}^1 \int_{-1}^1 (1+\xi^2)(1+\eta^2) \, d\xi \, d\eta = \frac{\rho abt}{4^3} \left[\xi + \frac{\xi^3}{3} \right]_{-1}^1 \left[\eta + \frac{\eta^3}{3} \right]_{-1}^1$$

$$m_{11} = \frac{\rho abt}{4^3} \left(2 + \frac{2}{3} \right) \left(2 + \frac{2}{3} \right) = \frac{\rho abt}{9} = 4 \frac{\rho abt}{36}$$

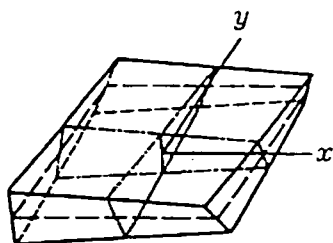
(continues)

11.3-7

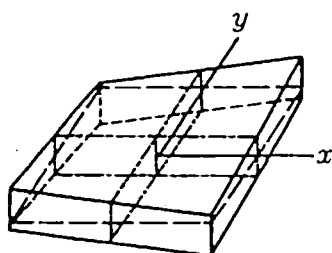


11.3-8

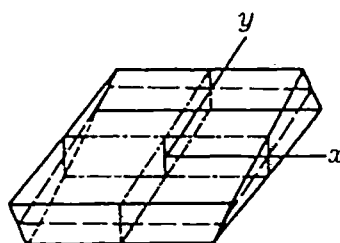
$$\{\underline{\kappa}_M\} = \begin{Bmatrix} \psi_{x,x} \\ \psi_{y,y} \\ \psi_{x,y} + \psi_{y,x} \\ \psi_x - w_{,x} \\ \psi_y - w_{,y} \end{Bmatrix} \begin{matrix} \text{curvature} \\ \text{curvature} \\ \text{twist} \\ \gamma_{zx} \\ \gamma_{yz} \end{matrix}$$



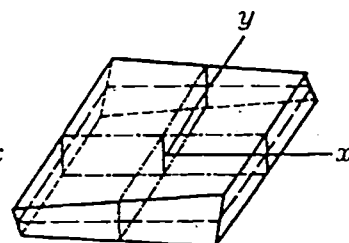
In-plane twist mode
 $w=0, \psi_x = -y, \psi_y = x$



w-hourglass mode
 $w=xy, \psi_x = \psi_y = 0$



ψ_x -hourglass mode
 $w=0, \psi_x = xy, \psi_y = 0$



ψ_y -hourglass mode
 $w=0, \psi_x = 0, \psi_y = xy$

For the above four modes, $\{\underline{\kappa}_M\}$ is

$$\begin{Bmatrix} 0 \\ 0 \\ -1+1=0 \\ x \\ -y \end{Bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -x \\ -y \end{Bmatrix} \begin{Bmatrix} y \\ 0 \\ x \\ 0 \\ xy \end{Bmatrix} \begin{Bmatrix} 0 \\ x \\ y \\ xy \\ 0 \end{Bmatrix} \begin{matrix} \uparrow \\ \text{bending} \\ \& \text{twist} \\ \downarrow \\ \text{transverse} \\ \text{shear} \\ \uparrow \end{matrix}$$

in-plane twist w-hourglass ψ_x -hourglass ψ_y -hourglass

Reduced integration: one Gauss pt. at $x=y=0$,
so all 4 above vectors are zero there,

Selective integration: one point for
shear terms (at $x=y=0$), four points
for bending terms. The four points
have nonzero x and y , and so
detect the curvature and twist
terms with the ψ_x and ψ_y hourglass
modes; they cease to be mechanisms.

11.3-8 (concluded)

$$m_{12} = \frac{pabt}{4^3} \int_{-1}^1 \int_{-1}^1 (1-\xi^2)(1-2\eta+\eta^2) d\xi d\eta = \frac{pabt}{4^3} \left[\xi - \frac{\xi^3}{3} \right]_{-1}^1 \left[\eta + \frac{\eta^3}{3} \right]_{-1}^1$$

$$m_{12} = \frac{pabt}{4^3} \left(2 - \frac{2}{3}\right) \left(2 + \frac{2}{3}\right) = \frac{pabt}{18} = 2 \frac{pabt}{36}$$

$$m_{13} = \frac{pabt}{4^3} \int_{-1}^1 \int_{-1}^1 (1-\xi^2)(1-\eta^2) d\xi d\eta = \frac{pabt}{4^3} \left[\xi - \frac{\xi^3}{3} \right]_{-1}^1 \left[\eta - \frac{\eta^3}{3} \right]_{-1}^1$$

$$m_{13} = \frac{pabt}{4^3} \left(2 - \frac{2}{3}\right) \left(2 - \frac{2}{3}\right) = \frac{pabt}{36}$$

$$[m] = \frac{pabt}{36} \begin{bmatrix} 4 & 2 & 1 & 2 \\ & 4 & 2 & 1 \\ & & 4 & 2 \\ \text{symm.} & & & 4 \end{bmatrix}$$

11.3-9

$\underline{d}_A = \underline{T} \underline{d}_B$ where the d.o.f. are

$$\underline{d}_A = [u_A \ v_A \ \theta_A]^T \text{ at } A$$

$$\underline{d}_B = [u_B \ v_B \ \theta_B]^T \text{ at node } B$$

The relational (transformation) matrix is

$$\underline{T} = \begin{bmatrix} 1 & 0 & -L \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Mass matrix for d.o.f. $\underline{d}_A$ is $\underline{m}_A = \begin{bmatrix} m & & \\ & m & \\ & & 0 \end{bmatrix}$

For d.o.f. $\underline{d}_B$ it is

$$\underline{m}_B = \underline{T}^T \underline{m}_A \underline{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -L & 0 & 1 \end{bmatrix} \begin{bmatrix} m & 0 & -mL \\ 0 & m & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{m}_B = \begin{bmatrix} m & 0 & -mL \\ 0 & m & 0 \\ -mL & 0 & mL^2 \end{bmatrix}$$

A diagonal form of $\underline{m}_B$, call it $\underline{m}_{BD}$, would yield from $\underline{m}_{BD} \ddot{\underline{d}}_B$ —

- No torque associated with θ_B if there is translational accel. $\ddot{u}_B$
- No force associated with u_B if there is angular acceleration $\ddot{\theta}_B$.

(a) Take node 5 as typical side node.

From Eqs. 6.4-1, $N_5 = \frac{1}{2}(1-\xi^2)(1-\eta)$

$$\int_{-1}^1 \int_{-1}^1 N_5^2 d\xi d\eta = \frac{1}{4} \int_{-1}^1 \int_{-1}^1 (1-\xi^2)^2 (1-\eta)^2 d\xi d\eta =$$

$$\frac{1}{4} \left[2 \left(1 - \frac{2}{3} + \frac{1}{5} \right) 2 \left(1 + \frac{1}{3} \right) \right] = \frac{8}{15} \frac{4}{3} = \frac{32}{45}$$

Take node 3 as typical corner node. From

$$\text{Eqs. 6.4-1, } N_3 = \frac{1}{4}(1+\xi+\eta+\xi\eta) - \frac{1}{4}(1+\xi-\eta-\xi\eta^2) - \frac{1}{4}(1-\xi^2+\eta-\xi^2\eta)$$

$$N_3 = \frac{1}{4}(-1+\xi^2+\eta^2+\xi\eta+\xi\eta^2+\xi^2\eta)$$

Since odd powers will integrate to zero, let's discard them in N_3^2 . Thus, what's left is

$$N_3^2 = \frac{1}{16}(1+\xi^4+\eta^4+3\xi^2\eta^2+\xi^2\eta^4+\xi^4\eta^2-2\xi^2-2\eta^2)$$

$$\int_{-1}^1 N_3^2 d\xi = \frac{1}{8} \left(\frac{8}{15} - \frac{4}{5}\eta^2 + \frac{4}{3}\eta^4 \right)$$

$$\int_{-1}^1 \int_{-1}^1 N_3^2 d\xi d\eta = \frac{1}{4} \left(\frac{8}{15} - \frac{4}{15} + \frac{4}{15} \right) = \frac{2}{15}$$

$$s = \sum_i m_{ii} = \rho t \frac{A}{4} \left(4 \frac{32}{45} + 4 \frac{2}{15} \right) = \rho t A \frac{38}{45}$$

$$m = \rho t A; \quad \frac{m}{s} = \frac{45}{38}$$

$$\text{side node } m_{ii} = \frac{45}{38} \left(\rho t \frac{A}{4} \frac{32}{45} \right) = \frac{8}{38} \rho A t = \frac{16}{76} m$$

$$\text{corner node } m_{ii} = \frac{45}{38} \left(\rho t \frac{A}{4} \frac{2}{15} \right) = \frac{1.5}{38} \rho A t = \frac{3}{76} m$$

(b) Take node 5 as typical side node.

$$\text{From Table 6.6-1, } N_5 = \frac{1}{2}(1-\xi^2)(\eta^2-\eta)$$

$$\int_{-1}^1 \int_{-1}^1 N_5^2 d\xi d\eta = \frac{1}{4} \left[2 \left(1 - \frac{2}{3} + \frac{1}{5} \right) 2 \left(\frac{1}{5} + \frac{1}{3} \right) \right] = \left(\frac{8}{15} \right)^2$$

Take node 1 as typical corner node. From Table 6.4-1, $N_1 = \frac{5\xi\eta}{4}(1-\xi)(1-\eta)$. Dropping odd powers, which will integrate to zero,

$$N_1^2 = \frac{1}{16}(\xi^2\eta^2+\xi^4\eta^2+\xi^2\eta^4+\xi^4\eta^4)$$

$$\int_{-1}^1 \int_{-1}^1 N_1^2 d\xi d\eta = \frac{1}{4} \left(\frac{1}{3} \frac{1}{3} + \frac{1}{5} \frac{1}{3} + \frac{1}{3} \frac{1}{5} + \frac{1}{5} \frac{1}{5} \right) = \frac{16}{225}$$

$$\text{Center node: } \int_{-1}^1 \int_{-1}^1 (1-\xi^2)^2 (1-\eta^2)^2 d\xi d\eta$$

$$= \left[2 \left(1 - \frac{2}{3} + \frac{1}{5} \right) \right]^2 = \left(\frac{16}{15} \right)^2$$

$$s = \sum_i m_{ii} = \rho t \frac{A}{4} \left[\left(\frac{8}{15} \right)^2 4 + \frac{16}{225} 4 + \left(\frac{16}{15} \right)^2 \right]$$

$$s = \rho t A (0.64), \quad m = \rho t A, \quad \frac{m}{s} = \frac{1}{0.64}$$

$$\text{side node } m_{ii} = \frac{1}{0.64} \left[\rho t \frac{A}{4} \left(\frac{8}{15} \right)^2 \right] = \rho t A (0.1111)$$

$$= \frac{m}{9} = \frac{4m}{36}$$

$$\text{corner node } m_{ii} = \frac{1}{0.64} \left[\rho t \frac{A}{4} \left(\frac{16}{225} \right) \right] = \rho t A (0.0278)$$

$$= \frac{m}{36}$$

$$\text{center node } m_{ii} = \frac{1}{0.64} \left[\rho t \frac{A}{4} \left(\frac{16}{15} \right)^2 \right] = \rho t A (0.444)$$

$$= \frac{16m}{36}$$

11.3-11

(a) For $\theta_1 = \theta_2$, $v = (1-\xi)v_1 + \xi v_2$, which is a linear function

Next consider a simply supported beam ($v_1 = v_2 = 0$) with $\theta_{21} = -\theta_{22}$.

$$\begin{array}{c} \text{---} L \text{---} \\ \text{---} \xi \text{ or } x = L\xi \end{array}$$

$$v = \frac{L}{2}(\xi - \xi^2)(-\theta_{22}) + \frac{L}{2}(-\xi + \xi^2)\theta_{22}$$

$$v = L(\xi^2 - \xi)\theta_{22} = L\left(\frac{x^2}{L^2} - \frac{x}{L}\right)\theta_{22}$$

$$\frac{dv}{dx} = L\left(\frac{2x}{L^2} - \frac{1}{L}\right)\theta_{22} \quad \frac{d^2v}{dx^2} = \frac{2}{L}\theta_{22} \quad (A)$$

Beam theory: consider constant curvature, $d^2v/dx^2 = c_1$

Then $dv/dx = c_1x + c_2$ and $v = c_1x^2/2 + c_2x + c_3$

Boundary conditions: $v = 0$ at $x = 0$, so $c_3 = 0$

$v = 0$ at $x = L$, so $c_2 = -c_1L/2$

$$\frac{(dv/dx)_{x=L}}{d^2v/dx^2} = \frac{c_1L + c_2}{c_1} = \frac{L}{2}$$

And from Eqs. (A),

$$\frac{(dv/dx)_{x=L}}{d^2v/dx^2} = L\left(\frac{2}{L} - \frac{1}{L}\right)\theta_{22} / (2/L)\theta_{22} = \frac{L}{2}$$

agree

$$(b) [N] = \begin{bmatrix} (1 - \frac{x}{L}) & (\frac{x}{2} - \frac{x^2}{2L}) & \frac{x}{L} & (-\frac{x}{2} + \frac{x^2}{2L}) \end{bmatrix}$$

$$[m] = \rho A \int_0^L [N]^T [N] dx$$

But tedious expansion and integration, and with $m = \rho AL$,

$$[m] = \frac{m}{120} \begin{bmatrix} 40 & 5L & 20 & -5L \\ 5L & L^2 & 5L & -L^2 \\ 20 & 5L & 40 & -5L \\ -5L & -L^2 & -5L & L^2 \end{bmatrix}$$

$$(c) s = 2 \left(40 \frac{m}{120} \right) = \frac{2m}{3}, \quad \frac{m}{s} = \frac{3}{2}$$

$$[m] = \frac{m}{120} \begin{bmatrix} 60 & \frac{3L^2}{2} & 60 & \frac{3L^2}{2} \\ 60 & \frac{3L^2}{2} & 60 & \frac{3L^2}{2} \end{bmatrix} = \frac{m}{80} \begin{bmatrix} 40 & L^2 & 40 & L^2 \end{bmatrix}$$

11.4-1

Let λ , $\Delta\lambda$, and amplitude $\underline{D}$ correspond to mode i .

$$(\underline{D} + \Delta\underline{D})^T (\underline{K} + \Delta\underline{K}) (\underline{D} + \Delta\underline{D}) = (\lambda + \Delta\lambda) (\underline{D} + \Delta\underline{D})^T (\underline{M} + \Delta\underline{M}) (\underline{D} + \Delta\underline{D})$$

$$\underline{D}^T \underline{K} \underline{D} + \underline{D}^T \underline{K} \Delta\underline{D} + \Delta\underline{D}^T \underline{K} \underline{D} + \underline{D}^T \Delta\underline{K} \underline{D} + \text{(higher terms)} = (\lambda \underline{D}^T \underline{M} \underline{D} + \Delta\lambda \underline{D}^T \underline{M} \underline{D}) + \lambda (\underline{D}^T \underline{M} \Delta\underline{D} + \Delta\underline{D}^T \underline{M} \underline{D} + \underline{D}^T \Delta\underline{M} \underline{D} + \text{higher terms})$$

But $\underline{D}^T \underline{K} \underline{D} = \lambda \underline{D}^T \underline{M} \underline{D}$ from Eq. 11.4-13.

Also $\underline{D}^T \underline{K} \Delta\underline{D} = \Delta\underline{D}^T \underline{K} \underline{D}$, $\underline{D}^T \underline{M} \Delta\underline{D} = \Delta\underline{D}^T \underline{M} \underline{D}$.

Thus, and dropping higher-order terms,

$$2 \Delta\underline{D}^T \underline{K} \underline{D} + \underline{D}^T \Delta\underline{K} \underline{D} = \Delta\lambda \underline{D}^T \underline{M} \underline{D} + \lambda (2 \Delta\underline{D}^T \underline{M} \underline{D} + \underline{D}^T \Delta\underline{M} \underline{D})$$

or

$$2 \Delta\underline{D}^T (\underbrace{\underline{K} - \lambda \underline{M}}_{\text{zero}}) \underline{D} + \underline{D}^T (\Delta\underline{K} - \lambda \Delta\underline{M}) \underline{D} = \Delta\lambda \underline{D}^T \underline{M} \underline{D}$$

$$\text{Finally } \Delta\lambda = \frac{\underline{D}^T (\Delta\underline{K} - \lambda \Delta\underline{M}) \underline{D}}{\underline{D}^T \underline{M} \underline{D}}$$

11, 4-2

(a) Exact $\begin{vmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{vmatrix} = 0$ satisfied by
result: $\lambda = 1$ & $\lambda = 6$.

Approximate, first mode:

$$\begin{bmatrix} 1.7 & 1.0 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \begin{Bmatrix} 1.7 \\ 1.0 \end{Bmatrix} = 3.98$$

$$\lambda_1 \approx \frac{3.98}{3.89} = 1.023$$

$$\begin{bmatrix} 1.7 & 1.0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1.7 \\ 1.0 \end{Bmatrix} = 3.89 \quad (\text{high, as expected})$$

Approximate, second mode:

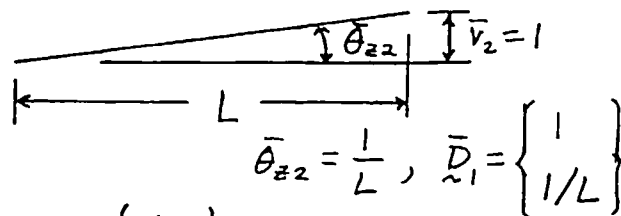
$$\begin{bmatrix} 1.2 & -2.0 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \begin{Bmatrix} 1.2 \\ -2.0 \end{Bmatrix} = 32.48$$

$$\lambda_2 \approx \frac{32.48}{5.44} = 5.97$$

$$\begin{bmatrix} 1.2 & -2.0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1.2 \\ -2.0 \end{Bmatrix} = 5.44 \quad (\text{low, as expected})$$

Approx. evals. more accurate than approx. evecs.

(b) $\tilde{K} = \frac{EI}{L^3} \begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}$, $\tilde{M} = \frac{\rho AL}{420} \begin{bmatrix} 156 & -22L \\ -22L & 4L^2 \end{bmatrix}$



$$\tilde{K} \bar{D}_1 = \frac{EI}{L^3} \begin{Bmatrix} 6 \\ -2L \end{Bmatrix}, \quad \bar{D}_1^T \tilde{K} \bar{D}_1 = \frac{4EI}{L^3}$$

$$\tilde{M} \bar{D}_1 = \frac{\rho AL}{420} \begin{Bmatrix} 134 \\ -18L \end{Bmatrix}, \quad \bar{D}_1^T \tilde{M} \bar{D}_1 = 116 \frac{\rho AL}{420}$$

$$\omega_1^2 \approx \frac{4}{116/420} \frac{EI}{\rho AL^4} = 14.48 \frac{EI}{\rho AL^4}$$

$$\omega_1 \approx 3.81 \left(\frac{EI}{\rho AL^4} \right)^{1/2} \quad [\text{exact } \omega_1 = 3.516 (EI/\rho AL^4)^{1/2}]$$

Better: use shape of cantilever beam under transverse tip load P :

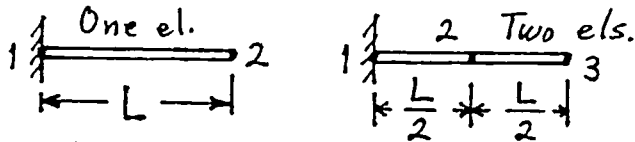
$$v = \frac{PL^3}{3EI}, \quad \theta = \frac{PL^2}{2EI} = \frac{3v}{2L} \quad \text{so use } \bar{D}_1 = \begin{Bmatrix} 1 \\ 3/2L \end{Bmatrix}$$

With foregoing $\tilde{K}$ and $\tilde{M}$,

$$\tilde{K} \bar{D}_1 = \frac{EI}{L^3} \begin{Bmatrix} 3 \\ 0 \end{Bmatrix}, \quad \bar{D}_1^T \tilde{K} \bar{D}_1 = \frac{3EI}{L^3} \quad \left\{ \begin{array}{l} \omega_1^2 \approx \frac{3(420)}{99} \frac{EI}{\rho AL^4} \end{array} \right.$$

$$\tilde{M} \bar{D}_1 = \frac{\rho AL}{420} \begin{Bmatrix} 123 \\ -16L \end{Bmatrix}, \quad \bar{D}_1^T \tilde{M} \bar{D}_1 = \frac{99 \rho AL}{420} \quad \left\{ \begin{array}{l} \omega_1^2 \approx 12.73 \frac{EI}{\rho AL^4}, \quad \omega_1 \approx 3.568 \left(\frac{EI}{\rho AL^4} \right)^{1/2} \end{array} \right.$$

11.4-3



(a) One el. $\left(\frac{AE}{L} - \omega_1^2 \frac{m}{3}\right) \bar{u}_2 = 0, \omega_1^2 = \frac{3AE}{mL}$

Two els. $\left(\frac{AE}{L/2} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \frac{m}{2} \omega^2 \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\right) \begin{Bmatrix} \bar{u}_2 \\ \bar{u}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$7\lambda^2 - 10\lambda + 1 = 0$, where $\lambda = \frac{m\omega^2 L}{24AE}$

$\lambda_1 = 0.1082, \omega_1^2 = 2.597 (AE/mL)$

(b) One el. $\left(\frac{AE}{L} - \omega_1^2 \frac{m}{2}\right) \bar{u}_2 = 0, \omega_1^2 = \frac{2AE}{mL}$

Two els. $\left(\frac{AE}{L/2} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \frac{m}{2} \omega^2 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right) \begin{Bmatrix} \bar{u}_2 \\ \bar{u}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$2\lambda^2 - 4\lambda + 1 = 0$, where $\lambda = \frac{m\omega^2 L}{8AE}$

$\lambda_1 = 0.2929, \omega_1^2 = 2.343 (AE/mL)$

(c) $\begin{Bmatrix} m \\ \bar{u} \end{Bmatrix} = \frac{1}{2} \left(\frac{m}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{m}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \frac{m}{12} \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$

One el. $\left(\frac{AE}{L} - \omega_1^2 \frac{5m}{12}\right) \bar{u}_2 = 0, \omega_1^2 = \frac{2.4AE}{mL}$

Two els. $\left(\frac{AE}{L/2} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \frac{m}{2} \omega^2 \begin{bmatrix} 10 & 1 \\ 1 & 5 \end{bmatrix}\right) \begin{Bmatrix} \bar{u}_2 \\ \bar{u}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$49\lambda^2 - 22\lambda + 1 = 0$, where $\lambda = \frac{m\omega^2 L}{48AE}$

$\lambda_1 = 0.2929, \omega_1^2 = 2.463 (AE/mL)$

(d) $\left(\frac{AE}{L} - \omega_1^2 m \frac{3-\beta}{6}\right) \bar{u}_2 = 0, \omega_1^2 = \frac{6AE}{(3-\beta)mL}$

From: $\omega_1^2 = \left(\frac{\pi}{2}\right)^2 \frac{AE}{mL}$

Equate these ω_1^2 values; thus

$\frac{6}{3-\beta} = \left(\frac{\pi}{2}\right)^2$ and $\beta = 0.568$

11.4-4

$$\text{Exact } \omega_1 = \frac{\pi}{2} \left(\frac{AE}{mL} \right)^{1/2} = 1.5708 \left(\frac{AE}{mL} \right)^{1/2}$$

frequencies * $(AE/mL)^{1/2}$ (from Prob. 11.4-3)

<u>N</u>	<u>consis.</u>	<u>lumped</u>	<u>ave.</u>
1	1.732	1.414	1.549
2	1.612	1.531	1.569

<u>% errors</u>			
<u>N</u>	<u>consis.</u>	<u>lumped</u>	<u>ave.</u>
1	10.3	-10.0	-1.4
2	2.6	-2.5	-0.1
F	3.96	4.00	14

where F is the factor of reduction of magnitude of % error in going from N=1 to N=2 elements

These F factors agree with the respective error estimates $O(h^2)$, $O(h^2)$, and $O(h^4)$.

11.4-5

(a) $m = \rho AL$. Let $\lambda = \frac{3L}{AE} \frac{m\omega^2}{30} = \frac{mL\omega^2}{10AE}$

The eigenvalue problem becomes

$$\begin{vmatrix} 7-4\lambda & -8-2\lambda & 1+\lambda \\ -8-2\lambda & 16(1-\lambda) & -8-2\lambda \\ 1+\lambda & -8-2\lambda & 7-4\lambda \end{vmatrix} = 0, \text{ from which}$$

$$5\lambda(5\lambda^2 - 36\lambda + 36) = 0 \quad \lambda_1 = 0, \omega_1 = 0$$

$$\lambda = \frac{36 \pm \sqrt{36^2 - 4(5)36}}{10} \quad \lambda_2 = 1.2, \omega_2 = 3.464 \sqrt{\frac{AE}{mL}}$$

$$\lambda_3 = 6.0, \omega_3 = 7.746 \sqrt{\frac{AE}{mL}}$$

(b) $m = \rho AL$. Let $\lambda = \frac{3L}{AE} \frac{m\omega^2}{6} = \frac{mL\omega^2}{2AE}$

The eigenvalue problem becomes

$$\begin{vmatrix} 7-\lambda & -8 & 1 \\ -8 & 16-4\lambda & -8 \\ 1 & -8 & 7-\lambda \end{vmatrix} = 0, \text{ from which}$$

$$4\lambda(\lambda^2 - 18\lambda + 72) = 0 \quad \lambda_1 = 0, \omega_1 = 0$$

$$\lambda = \frac{18 \pm \sqrt{18^2 - 4(72)}}{2} \quad \lambda_2 = 6, \omega_2 = 3.464 \sqrt{\frac{AE}{mL}}$$

$$\lambda_3 = 12, \omega_3 = 4.899 \sqrt{\frac{AE}{mL}}$$

(c) Rigid body translation

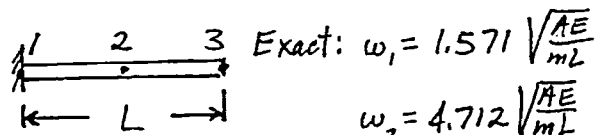
(d) Exact: $\omega_1 = 0, \omega_2 = 3.142 \sqrt{\frac{AE}{mL}}, \omega_3 = 6.283 \sqrt{\frac{AE}{mL}}$

Errors of (a): 0% Errors of (b): 0%

+10.3% +10.3%

+23.3% -22.0%

11.4-6



$$(a) \left(\frac{AE}{3L} \begin{bmatrix} 16 & -8 \\ -8 & 7 \end{bmatrix} - \frac{m\omega^2}{30} \begin{bmatrix} 16 & 2 \\ 2 & 4 \end{bmatrix} \right) \begin{Bmatrix} \bar{u}_2 \\ \bar{u}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Let } \lambda = \frac{mL\omega^2}{10AE}; \text{ then } \begin{vmatrix} 16(1-\lambda) & -8-2\lambda \\ -8-2\lambda & 7-4\lambda \end{vmatrix} = 0$$

$$15\lambda^2 - 52\lambda + 12 = 0 \quad \omega_1 = 1.577 \sqrt{\frac{AE}{mL}}$$

$$\lambda_1 = 0.248596$$

$$\lambda_2 = 3.21807 \quad \omega_2 = 5.673 \sqrt{\frac{AE}{mL}}$$

$$(b) \left(\frac{AE}{3L} \begin{bmatrix} 16 & -8 \\ -8 & 7 \end{bmatrix} - \frac{m\omega^2}{6} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{Bmatrix} \bar{u}_2 \\ \bar{u}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Let } \lambda = \frac{mL\omega^2}{2AE}; \text{ then } \begin{vmatrix} 16-4\lambda & -8 \\ -8 & 7-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 11\lambda + 12 = 0 \quad \omega_1 = 1.523 \sqrt{\frac{AE}{mL}}$$

$$\lambda_1 = 1.22800$$

$$\lambda_2 = 9.77200 \quad \omega_2 = 4.421 \sqrt{\frac{AE}{mL}}$$

$$(c) \left(\frac{AE}{3L} \begin{bmatrix} 16 & -8 \\ -8 & 7 \end{bmatrix} - \frac{m\omega^2}{3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{Bmatrix} \bar{u}_2 \\ \bar{u}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Let } \lambda = \frac{mL\omega^2}{AE}; \text{ then } \begin{vmatrix} 16-\lambda & -8 \\ -8 & 7-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 23\lambda + 48 = 0 \quad \omega_1 = 1.524 \sqrt{\frac{AE}{mL}}$$

$$\lambda_1 = 2.32122$$

$$\lambda_2 = 20.6788 \quad \omega_2 = 4.547 \sqrt{\frac{AE}{mL}}$$

$$(d) \{d\}^T [k] \{d\} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}^T \begin{bmatrix} 16 & -8 \\ -8 & 7 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} = 12 \quad \left(* \frac{AE}{3L} \right)$$

$$\text{Consistent: } \{d\}^T [m] \{d\} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}^T \begin{bmatrix} 16 & 2 \\ 2 & 4 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} = 40 \quad \left(* \frac{m}{30} \right)$$

$$\omega_1^2 = \frac{12/3}{40/30} \frac{AE}{mL}, \quad \omega_1 = 1.732 \sqrt{\frac{AE}{mL}}$$

HRZ-lumped:

$$\{d\}^T [k] \{d\} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}^T \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} = 8 \quad \left(* \frac{m}{6} \right)$$

$$\omega_1^2 = \frac{12/3}{8/6} \frac{AE}{mL}, \quad \omega_1 = 1.732 \sqrt{\frac{AE}{mL}}$$

Ad-hoc lumped:

$$\{d\}^T [m] \{d\} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} = 5 \quad \left(* \frac{m}{3} \right)$$

$$\omega_1^2 = \frac{12/3}{5/3} \frac{AE}{mL}, \quad \omega_1 = 1.549 \sqrt{\frac{AE}{mL}}$$

11.4-7

Exact $\omega_1 = 2.4674 [EI/\rho AL^4]^{1/2}$, where L is the half-length. In each part of this problem,

$$[k] = \frac{2EI}{2L} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \text{ and } ([k] - \omega^2 [m]) \begin{Bmatrix} \bar{\theta}_1 \\ \bar{\theta}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

(a) $[m] = [0]$, as no mass is associated with d.o.f. θ_1 and θ_2 . No solution ($\omega \rightarrow \infty$).

$$(a) [m] = \frac{\rho A(2L)}{420} \begin{bmatrix} 4(2L)^2 & -3(2L)^2 \\ -3(2L)^2 & 4(2L)^2 \end{bmatrix}$$

$$\text{Let } \lambda = \frac{2\rho AL^4 \omega^2}{105EI}, \text{ then } \begin{vmatrix} 2-4\lambda & 1+3\lambda \\ 1+3\lambda & 2-4\lambda \end{vmatrix} = 0$$

$$7\lambda^2 - 22\lambda + 3 = 0$$

$$\lambda_1 = 1/7$$

$$\lambda_2 = 21/7$$

$$\omega_1 = 2.739 [EI/\rho AL^4]^{1/2}$$

$$\omega_2 = 12.55 [EI/\rho AL^4]^{1/2}$$

(b) $[m] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. No solution (or ω 's = ∞)

$$(c) [m] = \frac{17.5(2\rho AL)(2L)^2}{2(210)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{Let } \lambda = \frac{\rho AL^4 \omega^2}{3EI}, \text{ then } \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$\lambda_1 = 1$$

$$\lambda_2 = 3$$

$$\omega_1 = 1.732 [EI/\rho AL^4]^{1/2}$$

$$\omega_2 = 3.000 [EI/\rho AL^4]^{1/2}$$

$$(d) [m] = \frac{\rho A(2L)}{78} \begin{bmatrix} (2L)^2 & 0 \\ 0 & (2L)^2 \end{bmatrix} = \frac{4\rho AL^3}{39} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{Let } \lambda = \frac{4\rho AL^4 \omega^2}{39EI}, \text{ then } \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$\lambda_1 = 1$$

$$\lambda_2 = 3$$

$$\omega_1 = 3.122 [EI/\rho AL^4]^{1/2}$$

$$\omega_2 = 5.408 [EI/\rho AL^4]^{1/2}$$

$$(e) [m] = \frac{\rho A(2L)}{120} (2L)^2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{\rho AL^3}{15} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\text{Let } \lambda = \frac{\rho AL^4 \omega^2}{15EI}, \text{ then } \begin{vmatrix} 2-\lambda & 1+\lambda \\ 1+\lambda & 2-\lambda \end{vmatrix} = 0$$

$$-6\lambda + 3 = 0, \lambda = \frac{1}{2}, \omega_1 = 2.739 [EI/\rho AL^4]^{1/2}$$

(Same as ω_1 of part (a) -- const. v_{xx} mode)

11.4-8

$$\text{Exact } \omega_1 = 3.516 [EI/\rho AL^4]^{1/2}$$

$$[k] = \frac{2EI}{L^3} \begin{bmatrix} 6 & -3L \\ -3L & 2L^2 \end{bmatrix}, \quad ([k] - \omega^2 [m]) \begin{Bmatrix} \bar{v}_2 \\ \bar{\theta}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Lumped $[m]$ solution:

$$[m] = \frac{\rho AL}{2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{Let } \lambda = \frac{\rho AL^4 \omega^2}{4EI}, \text{ then } \begin{vmatrix} 6-\lambda & -3L \\ -3L & 2L^2 \end{vmatrix} = 0$$

$$\lambda = \frac{3}{2}, \quad \omega_1 = 2.449 [EI/\rho AL^4]^{1/2}$$

(ω_2 is not provided)

Using $[m]$ from Prob. 11.3-11(b):

$$[m] = \frac{\rho AL}{120} \begin{bmatrix} 40 & -5L \\ -5L & L^2 \end{bmatrix}$$

$$\text{Let } \lambda = \frac{\rho AL^4 \omega^2}{240EI}, \text{ then } \begin{vmatrix} 6-40\lambda & -3L+5\lambda L \\ -3L+5\lambda L & (2-\lambda)L^2 \end{vmatrix} = 0$$

$$15\lambda^2 - 56\lambda + 3 = 0$$

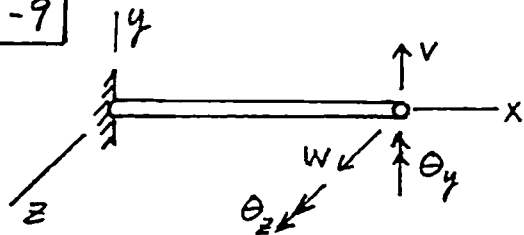
$$\lambda_1 = 0.054363$$

$$\lambda_2 = 3.6790$$

$$\omega_1 = 3.612 [EI/\rho AL^4]^{1/2}$$

$$\omega_2 = 29.714 [EI/\rho AL^4]^{1/2}$$

11.4-9



$$\{\bar{D}\} = \begin{Bmatrix} v \\ \theta_2 \\ w \\ \theta_2 \end{Bmatrix}$$

$$[M] = \begin{bmatrix} m_{11} & m_{12} & 0 & 0 \\ m_{12} & m_{22} & 0 & 0 \\ 0 & 0 & m_{11} & m_{12} \\ 0 & 0 & m_{12} & m_{22} \end{bmatrix}$$

Activate y-direction motion:

$$[M]\{\bar{D}_y\} = \begin{Bmatrix} m_{11}v + m_{12}\theta_2 \\ m_{12}v + m_{22}\theta_2 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{\bar{D}_y\} = \begin{Bmatrix} v \\ \theta_2 \\ 0 \\ 0 \end{Bmatrix}$$

Premultiply by z-direction motion, $\{\bar{D}_z\} = \begin{Bmatrix} 0 \\ 0 \\ w \\ \theta_2 \end{Bmatrix}$

$$\text{Get } \{\bar{D}_z\}^T ([M]\{\bar{D}_y\}) = 0$$

11.4-10

Exact: $c = \pi^2/4 = 2.4674$

(a) $p=2$ $c = \frac{3.000(1) - 2.597(4)}{1-4} = 2.463$

(b) $p=2$ $c = \frac{2.000(1) - 2.343(4)}{1-4} = 2.457$

(c) $p=4$ $c = \frac{2.400(1) - 2.463(16)}{1-16} = 2.467$

11.5-1

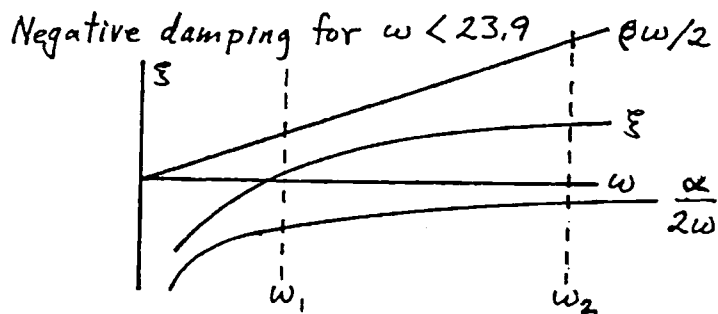
(a) $\xi_1 = 0.03 \quad \omega_1 = 2\pi(5) = 10$
 $\xi_2 = 0.20 \quad \omega_2 = 2\pi(15) = 30$

Apply Eqs. 11.5-3:

$$\beta = 2 \frac{0.2(30\pi) - 0.03(10\pi)}{(30\pi)^2 - (10\pi)^2} = 0.004536$$

$$\alpha = 2(10\pi)(30\pi) \frac{0.03(30\pi) - 0.2(10\pi)}{(30\pi)^2 - (10\pi)^2} = -2.592$$

(b) $\xi = \frac{\alpha}{2\omega} + \frac{\beta\omega}{2} = -\frac{1.296}{\omega} + 0.002268\omega$



11.6-1

(a) Work out $[\tilde{T}]^T ([\tilde{K}][\tilde{T}])$.

$$\begin{bmatrix} \tilde{K}_{mm} & \tilde{K}_{ms} \\ \tilde{K}_{ms}^T & \tilde{K}_{ss} \end{bmatrix} \begin{bmatrix} \tilde{I} \\ -\tilde{K}_{ss}^{-1} \tilde{K}_{ms}^T \end{bmatrix} = \begin{bmatrix} \tilde{K}_{mm} - \tilde{K}_{ms} \tilde{K}_{ss}^{-1} \tilde{K}_{ms}^T \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \tilde{I} & -\tilde{K}_{ms} \tilde{K}_{ss}^{-1} \end{bmatrix} \begin{bmatrix} \tilde{I} \\ -\tilde{K}_{ss}^{-1} \tilde{K}_{ms}^T \end{bmatrix} = \tilde{K}_{mm} - \tilde{K}_{ms} \tilde{K}_{ss}^{-1} \tilde{K}_{ms}^T$$

$\tilde{K}_{ss}^{-T} = \tilde{K}_{ss}^{-1}$ by symmetry

Same result as given by "static condensation".

(b)

$$\begin{bmatrix} \tilde{I} & -\tilde{K}_{ms} \tilde{K}_{ss}^{-T} \end{bmatrix} \begin{bmatrix} \tilde{M}_{mm} & \tilde{M}_{ms} \\ \tilde{M}_{ms}^T & \tilde{M}_{ss} \end{bmatrix} \begin{bmatrix} \tilde{I} \\ -\tilde{K}_{ss}^{-1} \tilde{K}_{ms}^T \end{bmatrix} =$$

same

$$\begin{bmatrix} \tilde{I} & -\tilde{K}_{ms} \tilde{K}_{ss}^{-1} \end{bmatrix} \begin{bmatrix} \tilde{M}_{mm} - \tilde{M}_{ms} \tilde{K}_{ss}^{-1} \tilde{K}_{ms}^T \\ \tilde{M}_{ms}^T - \tilde{M}_{ss} \tilde{K}_{ss}^{-1} \tilde{K}_{ms}^T \end{bmatrix} =$$

$$\tilde{M}_{mm} - \tilde{M}_{ms} \tilde{K}_{ss}^{-1} \tilde{K}_{ms}^T - \tilde{K}_{ms} \tilde{K}_{ss}^{-1} \tilde{M}_{ms}^T + \tilde{K}_{ms} \tilde{K}_{ss}^{-1} \tilde{M}_{ss} \tilde{K}_{ss}^{-1} \tilde{K}_{ms}^T$$

11.6-2

$$(a) \quad \begin{bmatrix} \tilde{F}_{mm} & \tilde{F}_{ms} \\ \tilde{F}_{ms}^T & \tilde{F}_{ss} \end{bmatrix} \begin{Bmatrix} \tilde{R}_m \\ \tilde{R}_s \end{Bmatrix} = \begin{Bmatrix} \tilde{D}_m \\ \tilde{D}_s \end{Bmatrix}$$

With $\tilde{R}_s = 0$, the upper partition yields

$$\tilde{R}_m = \tilde{F}_{mm}^{-1} \tilde{D}_m. \text{ Therefore}$$

$$\begin{Bmatrix} \tilde{D}_m \\ \tilde{D}_s \end{Bmatrix} = \begin{bmatrix} \tilde{F}_{mm} \\ \tilde{F}_{ms}^T \end{bmatrix} \begin{Bmatrix} \tilde{R}_m \end{Bmatrix} = \begin{bmatrix} \tilde{F}_{mm} \\ \tilde{F}_{ms}^T \end{bmatrix} [\tilde{F}_{mm}]^{-1} \tilde{D}_m$$

$$\begin{Bmatrix} \tilde{D}_m \\ \tilde{D}_s \end{Bmatrix} = [\tilde{T}_F] \tilde{D}_m, \text{ where } [\tilde{T}_F] = \begin{bmatrix} \tilde{I} \\ \tilde{F}_{ms}^T \tilde{F}_{mm}^{-1} \end{bmatrix}$$

(b)

$$\text{We must show that } \tilde{F}_{ms}^T \tilde{F}_{mm}^{-1} = -\tilde{K}_{ss}^{-1} \tilde{K}_{ms}^T$$

The product of stiffness and flexibility matrices must be a unit matrix.

$$\begin{bmatrix} \tilde{K}_{mm} & \tilde{K}_{ms} \\ \tilde{K}_{ms}^T & \tilde{K}_{ss} \end{bmatrix} \begin{bmatrix} \tilde{F}_{mm} & \tilde{F}_{ms} \\ \tilde{F}_{ms}^T & \tilde{F}_{ss} \end{bmatrix} = \begin{bmatrix} \tilde{I} & \tilde{0} \\ \tilde{0} & \tilde{I} \end{bmatrix}$$

$$\text{Row 2 times column 1 is } \tilde{K}_{ms}^T \tilde{F}_{mm} + \tilde{K}_{ss} \tilde{F}_{ms} = 0$$

$$\text{Premult. by } \tilde{K}_{ss}^{-1} : \tilde{K}_{ss}^{-1} \tilde{K}_{ms}^T \tilde{F}_{mm} = -\tilde{F}_{ms}^T$$

$$\text{Postmult. by } \tilde{F}_{mm}^{-1} : \tilde{K}_{ss}^{-1} \tilde{K}_{ms}^T = -\tilde{F}_{ms}^T \tilde{F}_{mm}^{-1}$$

(c) Consider as many load vectors as there are master d.o.f., & let each load vector consist of a single unit load on a master d.o.f.

Also set $\tilde{R}_s = 0$.

$$\begin{bmatrix} \tilde{K}_{mm} & \tilde{K}_{ms} \\ \tilde{K}_{ms}^T & \tilde{K}_{ss} \end{bmatrix} \begin{Bmatrix} \tilde{D}_m \\ \tilde{D}_s \end{Bmatrix} = \begin{Bmatrix} \tilde{R}_m \\ \tilde{0} \end{Bmatrix} = \begin{bmatrix} \tilde{I} \\ \tilde{0} \end{bmatrix} \quad \begin{array}{l} \text{Solve; note} \\ \text{that} \\ [\tilde{F}] = [\tilde{K}]^{-1} \end{array}$$

$$\begin{Bmatrix} \tilde{D}_m \\ \tilde{D}_s \end{Bmatrix} = \begin{bmatrix} \tilde{F}_{mm} & \tilde{F}_{ms} \\ \tilde{F}_{ms}^T & \tilde{F}_{ss} \end{bmatrix} \begin{bmatrix} \tilde{I} \\ \tilde{0} \end{bmatrix} = \begin{bmatrix} \tilde{F}_{mm} \\ \tilde{F}_{ms}^T \end{bmatrix}$$

The physical meaning of $[\tilde{F}_{mm}]$ is that each of its columns represents the displacements of master d.o.f. produced by a unit load on one master d.o.f.

(d) Usually $m \ll s$, so $[\tilde{F}_{mm}]$ is much smaller than $[\tilde{K}_{ss}]$ -- cheaper to invert.

11.6-3

$$[\underline{K}] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}, \quad [\underline{M}] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

$$K_{ss} = k, K_{ms} = -k, [\underline{I}] = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \begin{aligned} [\underline{I}]^T [\underline{K}] [\underline{I}] &= 0 \\ [\underline{I}]^T [\underline{M}] [\underline{I}] &= 2m \end{aligned}$$

$$[0 - \omega^2(2m)] \bar{u}_1 = 0$$

Hence $\omega = 0$; the rigid body mode; OK.

11.6-4

(a) With c a constant, $\frac{M_{11}}{K_{11}} = c \frac{156}{12}$

and $\frac{M_{22}}{K_{22}} = c \frac{4L^2}{4L^2} = c \cdot \frac{M_{11}}{K_{11}} > \frac{M_{22}}{K_{22}}$, therefore

the choice is proper.

$$(b) [\tilde{T}] = \begin{bmatrix} -\frac{L^3}{12EI} & \frac{6EI}{L^2} \\ & 1 \end{bmatrix} = \begin{bmatrix} -\frac{L}{2} \\ 1 \end{bmatrix}$$

$$[\tilde{T}]^T [\tilde{K}] [\tilde{T}] = \begin{bmatrix} -\frac{L}{2} & 1 \end{bmatrix} \frac{EI}{L^3} \begin{bmatrix} -6L+6L \\ 3L^2+4L^2 \end{bmatrix} = \frac{EI}{L}$$

$$[\tilde{T}]^T [\tilde{M}] [\tilde{T}] = \begin{bmatrix} -\frac{L}{2} & 1 \end{bmatrix} \frac{m}{420} \begin{bmatrix} -78L-13L \\ 6.5L^2+4L^2 \end{bmatrix} = \frac{14mL^2}{105}$$

$$\left(\frac{EI}{L} - \omega^2 \frac{14mL^2}{105} \right) \bar{\theta}_2 = 0, \quad \omega^2 = 7.5 \frac{EI}{mL^3}$$

(c) Using the first recovery method (Eq. 11.6-3)

$$\begin{Bmatrix} \bar{v}_1 \\ \bar{\theta}_2 \end{Bmatrix} = [\tilde{T}] \bar{\theta}_2, \quad \bar{v}_1 = -\frac{L}{2} \bar{\theta}_2$$

$$\omega_1^2 = \frac{\frac{EI}{L^3} \begin{bmatrix} -\frac{L}{2} & 1 \end{bmatrix} \begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} -L/2 \\ 1 \end{Bmatrix}}{\frac{m}{420} \begin{bmatrix} -\frac{L}{2} & 1 \end{bmatrix} \begin{bmatrix} 156 & -13L \\ -13L & 4L^2 \end{bmatrix} \begin{Bmatrix} -L/2 \\ 1 \end{Bmatrix}}$$

$$\omega_1^2 = \frac{420EI}{mL^3} \frac{L^2}{56L} = 7.5 \frac{EI}{mL^3} \quad (\text{no improvement})$$

Using the 2nd recovery method (Eq. 11.6-6),

$$\bar{v}_1 = - \left[\frac{12EI}{L^3} - 7.5 \frac{EI}{mL^3} \frac{156m}{420} \right]^{-1} \left[\frac{6EI}{L^2} - 7.5 \frac{EI}{mL^3} \left(-\frac{13mL}{420} \right) \right] \bar{\theta}_2$$

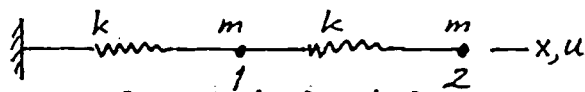
$$v_1 = - \left[\frac{L^3}{12} (1 - 2.186) \right]^{-1} \left[\frac{EI}{L^2} (6 + 0.232) \right] \bar{\theta}_2$$

$$\bar{v}_1 = -0.67636L \quad \text{for } \bar{\theta}_2 = 1$$

$$\omega_1^2 = \frac{\frac{EI}{L^3} \begin{bmatrix} -0.6764L & 1 \end{bmatrix} \begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} -0.6764L \\ 1 \end{Bmatrix}}{\frac{m}{420} \begin{bmatrix} -0.6764L & 1 \end{bmatrix} \begin{bmatrix} 156 & -13 \\ -13 & 4L^2 \end{bmatrix} \begin{Bmatrix} -0.6764L \\ 1 \end{Bmatrix}}$$

$$\omega_1^2 = \frac{420EI}{mL^2} \frac{1.373L^2}{92.95} = 6.205 \frac{EI}{mL^3}$$

11.6-5



$$\left(k \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} - \omega^2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \right) \begin{Bmatrix} \bar{u}_2 \\ \bar{u}_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

where, for convenience, d.o.f. are ordered so form of Eq. 11.6-3 need not be changed.

D.o.f. $\bar{u}_1$ has the smaller M_{ii}/K_{ii} , so it is slave. From Eq. 11.6-3, $[I] = \begin{Bmatrix} 1.0 \\ 0.5 \end{Bmatrix}$

With $k=1$ and $m=2$, Eqs. 11.6-4 yield

$$[1.0, 0.5] \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} 1.0 \\ 0.5 \end{Bmatrix} = 0.5$$

$$[1.0, 0.5] \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} 1.0 \\ 0.5 \end{Bmatrix} = 2.5$$

$$(0.5 - 2.5\omega_1^2)\bar{u}_2 = 0, \omega_1^2 = 0.200, \omega_1 = 0.4472$$

(The full 2 by 2 system yields $\omega_1 = 0.4370$)

For (say) $\bar{u}_2 = 1$, Eq. 11.6-3 yields $\bar{u}_1 = 0.5$.

Then Rayleigh quotient yields

$$\omega_1^2 = \frac{[1.0, 0.5] \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} 1.0 \\ 0.5 \end{Bmatrix}}{[1.0, 0.5] \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} 1.0 \\ 0.5 \end{Bmatrix}} = 0.200 \text{ as before}$$

For (say) $\bar{u}_2 = 1$, Eq. 11.6-6 yields

$$\bar{u}_1 = -[2.0 - 0.2(2)]^{-1}[-1.0 - 0.2(0)](1.0) = 0.625$$

Then Rayleigh quotient yields

$$\omega_1^2 = \frac{[1.0, 0.625] \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} 1.0 \\ 0.625 \end{Bmatrix}}{[1.0, 0.625] \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} 1.0 \\ 0.625 \end{Bmatrix}} \quad \omega_1^2 = 0.1910$$

$$\omega_1 = 0.4370$$

11.7-1

With $\underline{C}$ from Eq. 11.2-12, the operations $\underline{\phi}^T \underline{C} \underline{\phi}$ yield the damping in Eq. 11.7-6, i.e.

$$\underline{\phi}^T \underline{C} \underline{\phi} = \begin{bmatrix} 2\xi_1 \omega_1 & & \\ & 2\xi_2 \omega_2 & \\ & & \ddots \\ & & & 2\xi_n \omega_n \end{bmatrix}$$

Premultiply by $\underline{\phi}^{-T}$; postmultiply by $\underline{\phi}^{-1}$

$$\underline{C} = \underline{\phi}^{-T} \begin{bmatrix} 2\xi_1 \omega_1 & & \\ & 2\xi_2 \omega_2 & \\ & & \ddots \\ & & & 2\xi_n \omega_n \end{bmatrix} \underline{\phi}^{-1}$$

11.7-2

Let $\bar{\underline{D}}_i^*$ be a vector before normalization.

Evaluate c in $(\bar{\underline{D}}_i^*)^T \underline{M} \bar{\underline{D}}_i^* = c$

Scaled vector $\bar{\underline{D}}_i = \frac{1}{\sqrt{c}} \bar{\underline{D}}_i^*$ will yield $\bar{\underline{D}}_i^T \underline{M} \bar{\underline{D}}_i = 1$

From Prob. 11.4-2a, $\omega_1^2 = 1$, $\omega_2^2 = 6$, and $c = \sqrt{5}$

Now use Eqs. 11.7-5 and 11.7-6

$$[\underline{\Phi}] = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$$

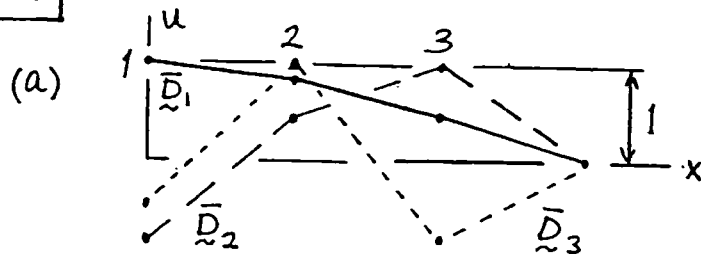
$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \frac{1}{\sqrt{5}} \begin{Bmatrix} 2R_1 + R_2 \\ R_1 - 2R_2 \end{Bmatrix}$$

Eq. 13.6-5 yields

$$\ddot{z}_1 + z_1 = (2R_1 + R_2)/\sqrt{5}$$

$$\ddot{z}_2 + 6z_2 = (R_1 - 2R_2)/\sqrt{5}$$

11.7-3



$\underline{\underline{M}} = m \underline{\underline{I}}$, so for $\underline{\underline{M}}$ -matrix orthogonality,

$$\underline{\underline{D}}_1^T \underline{\underline{D}}_2 = -0.802 + 0.802(0.445) + 0.445(1) = 0$$

$$\underline{\underline{D}}_1^T \underline{\underline{D}}_3 = -0.445 + 0.802(1) + 0.445(-0.802) = 0$$

$$\underline{\underline{D}}_2^T \underline{\underline{D}}_3 = -0.802(-0.445) + 0.445(1) - 0.802 = 0$$

$$\underline{\underline{K}} = k \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad \underline{\underline{K}} \underline{\underline{D}}_1 = k \begin{Bmatrix} 0.198 \\ 0.159 \\ 0.088 \end{Bmatrix},$$

$$\underline{\underline{K}} \underline{\underline{D}}_2 = k \begin{Bmatrix} -1.247 \\ 0.692 \\ 1.555 \end{Bmatrix}, \quad \underline{\underline{K}} \underline{\underline{D}}_3 = k \begin{Bmatrix} -1.445 \\ 3.247 \\ -2.604 \end{Bmatrix}, \quad \underline{\underline{D}}_i^T \underline{\underline{K}} \underline{\underline{D}}_j = \begin{cases} \text{values above} & i=j \\ 0 & i \neq j \end{cases}$$

(b) Must scale $\underline{\underline{D}}_1$ and $\underline{\underline{D}}_2$ according to Eq. 11.7-1, where $\underline{\underline{M}}$ is a unit matrix in this problem.

$$c_1^2 \underline{\underline{D}}_1^T \underline{\underline{I}} \underline{\underline{D}}_1 = c_1^2 (1.841) = 1; \quad c_1 = 0.737$$

$$c_2^2 \underline{\underline{D}}_2^T \underline{\underline{I}} \underline{\underline{D}}_2 = c_2^2 (1.841) = 1; \quad c_2 = 0.737$$

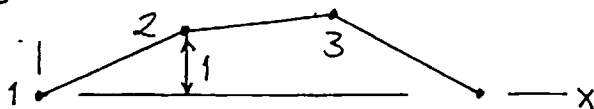
The rescaled eigenvectors are

$$\underline{\underline{D}}_1 = \begin{Bmatrix} 0.737 \\ 0.591 \\ 0.328 \end{Bmatrix}, \quad \underline{\underline{D}}_2 = \begin{Bmatrix} -0.591 \\ 0.328 \\ 0.737 \end{Bmatrix}$$

Eq. 11.7-4: $\begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \underline{\underline{D}}_1(1) + \underline{\underline{D}}_2 z_2$

First def. yields $0 = 0.737 - 0.591 z_2$
 $z_2 = 1.247$

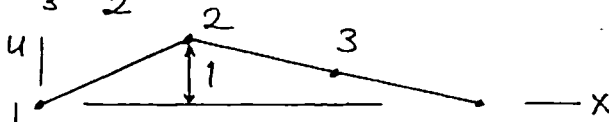
$$\therefore u_3 = 0.328 + 0.737(1.247) = 1.247$$



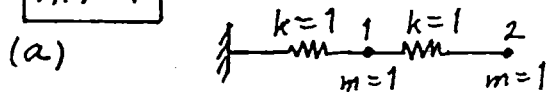
(c) For $u_1 = 0$ and $u_2 = 1$, from Eq. 11.6-2,

$$\underline{\underline{D}}_s = u_3 = -\underline{\underline{K}}_{ss}^{-1} \underline{\underline{K}}_{ms}^T \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = -\left(\frac{1}{2}\right) \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$u_3 = \frac{1}{2}$$



11.7-4



$$\left(\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Solution of this eigen problem yields

$$\lambda_1 = 0.381966, \quad \omega_1 = 0.618034$$

$$\lambda_2 = 2.61803, \quad \omega_2 = 1.618034$$

Eigenvectors, respectively un-normalized and normalized, are

$$\begin{Bmatrix} 1 \\ 1.61803 \end{Bmatrix} \rightarrow \begin{Bmatrix} 0.52573 \\ 0.85065 \end{Bmatrix}, \quad \begin{Bmatrix} 1 \\ -0.618034 \end{Bmatrix} \rightarrow \begin{Bmatrix} 0.85065 \\ -0.52573 \end{Bmatrix}$$

$$\text{Hence } [\phi] = \begin{bmatrix} 0.52573 & 0.85065 \\ 0.85065 & -0.52573 \end{bmatrix}$$

Initial conditions, using Eq. 11.7-4, are

$$\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix}_0 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix}_0 = [\phi]^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.85065 \\ -0.52573 \end{Bmatrix}$$

With $\xi_i = p_i = 0$, Eq. 11.7-6 has the solution

$$z_i = A_i \sin \omega_i t + B_i \cos \omega_i t$$

$B_i = 0$ because $z_i = 0$ at $t = 0$.

Next, $\dot{z}_i = A_i \omega_i \cos \omega_i t$, and at $t = 0$

$$\dot{z}_1 = 0.85065 = A_1 (0.618) \quad \left\{ \begin{array}{l} A_1 = 1.3764 \end{array} \right.$$

$$\dot{z}_2 = -0.52573 = A_2 (1.618) \quad \left\{ \begin{array}{l} A_2 = -0.3249 \end{array} \right.$$

$$z_1 = 1.3764 \sin 0.618 t$$

$$z_2 = -0.3249 \sin 1.618 t$$

By Eq. 11.7-4, $\{D\} = [\phi]^T \{z\}$ yields

$$\{u_2\} = \begin{Bmatrix} 0.618 t - 0.2764 \sin 1.618 t \\ 1.171 \sin 0.618 t + 0.1708 \sin 1.618 t \end{Bmatrix}$$

	t=0	t=1	t=2	t=3	t=4	t=5
u_1	0	.1432	.7095	.9684	.3972	-.2315
u_2	0	.0101	.1790	.9552	.7589	.2262

(c) Compare greatest magnitudes, regardless of the times at which they appear:

$$\frac{0.7236}{0.7236 + 0.2764} = 0.7236 \quad (27.6\% \text{ error in } u_1)$$

$$\frac{1.171}{1.171 + 0.1708} = 0.8727 \quad (12.7\% \text{ error in } u_2)$$

$$(b) \ddot{u}_1 = -0.276 \sin 0.618 t + 0.7236 \sin 1.618 t$$

$$\ddot{u}_2 = -0.447 \sin 0.618 t - 0.447 \sin 1.618 t$$

$$[M] \{\ddot{D}\} = \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix}$$

$$[K] \{D\} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0.276 \sin 0.618 t - 0.7236 \sin 1.618 t \\ 0.447 \sin 0.618 t + 0.447 \sin 1.618 t \end{Bmatrix}$$

$$[M] \{\ddot{D}\} + [K] \{D\} = \{0\} \quad \checkmark$$

11.7-5

From Eq. 11.7-6, with constant loads, $\ddot{Z}_i + \omega_i^2 Z_i = P_i$. Hence

$$Z_i = A_i \sin \omega_i t + B_i \cos \omega_i t + \frac{P_i}{\omega_i^2}, \quad \dot{Z}_i = A_i \omega_i \cos \omega_i t - B_i \omega_i \sin \omega_i t$$

At rest and not deformed at $t=0$, so $A_i = 0$, $B_i = -\frac{P_i}{\omega_i^2}$

$$\text{Thus } Z_i = \frac{P_i}{\omega_i^2}(1 - \cos \omega_i t) \quad \text{and} \quad \ddot{Z}_i = P_i \cos \omega_i t$$

$$\text{From Prob. 11.7-4, } [\phi] = \begin{bmatrix} 0.5257 & 0.8507 \\ 0.8507 & -0.5257 \end{bmatrix} \quad \text{and} \quad \begin{matrix} \omega_1 = 0.618 \\ \omega_2 = 1.618 \end{matrix}$$

$$\text{For } \{R\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}, \quad \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = [\phi]^T \{R\} = \begin{Bmatrix} 0.8507 \\ -0.5257 \end{Bmatrix}$$

$$\text{(a) First mode: } \frac{P_1}{\omega_1^2} = 2.227, \quad \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0.5257 \\ 0.8507 \end{Bmatrix} 2.227 (1 - \cos 0.618 t)$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 1.171 \\ 1.895 \end{Bmatrix} (1 - \cos 0.618 t)$$

	<u>t=2</u>	<u>t=4</u>	<u>t=6</u>	<u>t=8</u>	<u>t=10</u>
u_1	0.786	2.089	2.159	0.902	0.006
u_2	1.272	3.379	3.492	1.459	0.010

$$\text{(b) Second mode: } \frac{P_2}{\omega_2^2} = -0.2008, \quad \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0.8507 \\ -0.5257 \end{Bmatrix} (-0.2008) (1 - \cos 1.618 t)$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} -0.1708 \\ 0.1056 \end{Bmatrix} (1 - \cos 1.618 t)$$

	<u>t=2</u>	<u>t=4</u>	<u>t=6</u>	<u>t=8</u>	<u>t=10</u>
u_1	-0.341	-0.003	-0.335	-0.012	-0.323
u_2	0.211	0.002	0.207	0.011	0.200

Combine with first mode results to get final results:

	<u>t=2</u>	<u>t=4</u>	<u>t=6</u>	<u>t=8</u>	<u>t=10</u>
u_1	0.445	2.086	1.824	0.890	-0.317
u_2	1.483	3.381	3.699	1.470	0.210

(continues)

11.7-5 (concluded)

(c) For mode 1, $[K]\{D\} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0.447 \\ 0.724 \end{Bmatrix} (1 - \cos 0.618t)$

$$\{R\} - [M]\{\ddot{D}\} - [K]\{D\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} - \begin{Bmatrix} 0.447 \\ 0.724 \end{Bmatrix} \cos 0.618t - \begin{Bmatrix} 0.447 \\ 0.724 \end{Bmatrix} (1 - \cos 0.618t) \\ = \begin{Bmatrix} -0.447 \\ 0.276 \end{Bmatrix}$$

Eq. 11.7-9: $e(t) = 0.526$ (not small - indicates significant error)

For mode 2 (alone), $[K]\{D\} = \begin{Bmatrix} -0.447 \\ 0.276 \end{Bmatrix} (1 - \cos 1.618t)$

$$[M]\{\ddot{D}\} = \begin{Bmatrix} 0.447 \\ -0.276 \end{Bmatrix} \cos 1.618t$$

Using modes 1 and 2, $\{R\} - [M]\{\ddot{D}\} - [K]\{D\} = 0$ and $e(t) = 0$

(d) $[M][\Phi][\Phi]^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 0.5257 \\ 0.8507 \end{Bmatrix} [0.5257 \ 0.8507] = \begin{bmatrix} 0.2764 & 0.4472 \\ 0.4472 & 0.7273 \end{bmatrix}$

$\{R\}_{approx}^{ext} = \begin{Bmatrix} 0.2764 \\ 0.4472 \end{Bmatrix}$. From Eq. 11.7-11,

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \{\Delta D\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} - \{R\}_{approx}^{ext}, \quad \{\Delta D\} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} -0.4472 \\ 0.2727 \end{Bmatrix} = \begin{Bmatrix} -0.1745 \\ 0.0982 \end{Bmatrix}$$

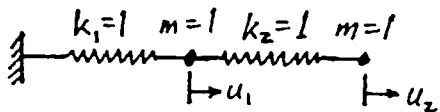
Add to results of part (a), to obtain:

	<u>t=2</u>	<u>t=4</u>	<u>t=6</u>	<u>t=8</u>	<u>t=10</u>
u_1	0.612	1.914	1.984	0.728	-0.169
u_2	1.370	3.477	3.590	1.557	0.108

Using the static correction, $[K]\{\Delta D\} = \begin{Bmatrix} -0.447 \\ 0.273 \end{Bmatrix}$

Including these terms in the numerator of Eq. 11.7-9, for mode 1 alone (as in the first part of part (c) above) gives $e(t) = 0$ (except for small rounding error).

11.8-1



Initial conditions: $u_1 = u_2 = \dot{u}_1 = 0, \dot{u}_2 = 1$

(a) Say $\{w_1\} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$ in Eq. 11.8-4. Then

Eq. 11.8-5 gives

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1^* \\ u_2^* \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} u_1^* \\ u_2^* \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} u_1^{**} \\ u_2^{**} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1-1 \\ 1-0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$[W] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ (yields original system)}$$

(b) Say $\{w_1\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$ in Eq. 11.8-4. Then

Eq. 11.8-5 gives

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1^* \\ u_2^* \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}, \quad \begin{Bmatrix} u_1^* \\ u_2^* \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

$$\begin{Bmatrix} u_1^{**} \\ u_2^{**} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1-0 \\ 2-2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

$$[W] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ (yields original system but with d.o.f. labels interchanged)}$$

(c) $[W^*] = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$; normalized $[W] = \frac{1}{\sqrt{5}} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$

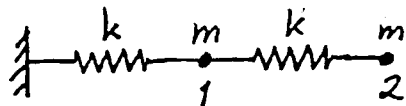
Eq. 11.8-6 becomes

$$\ddot{y} + \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} y = 0, \text{ or } \ddot{y} + \frac{2}{5} y = 0$$

Let $y = y_0 \sin \omega t$, this yields $\omega = \sqrt{\frac{2}{5}} = 0.6325$

(full system gives $\omega = 0.618$). The approximate solution amounts to use of an assumed mode shape in the Rayleigh quotient.

11.9-1



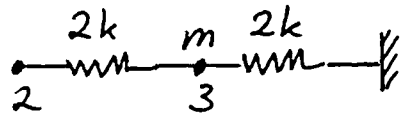
is substructure 1:

$$[\tilde{K}]_1 = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \end{matrix} \quad \psi = -\frac{1}{2}(-1) = \frac{1}{2}, \quad \begin{bmatrix} \tilde{\psi} \\ \tilde{I} \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

Mode 1, with node 2 fixed, is $(2 - \omega^2)u_1 = 0$, $\omega^2 = \sqrt{2}$, $u_1 = 1$ (say)

$$[\tilde{W}]_1 = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}, \quad [\tilde{W}]_1^T [\tilde{K}]_1 [\tilde{W}]_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{matrix} a_1 \\ u_2 \end{matrix}$$

Elect to associate m at node 2 with substructure 1.



is substructure 2:

$$[\tilde{W}]_1^T [\tilde{M}]_1 [\tilde{W}]_1 = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 5/4 \end{bmatrix} \begin{matrix} a_1 \\ u_2 \end{matrix}$$

From substructure 1, double stiffness, reorder $[\tilde{W}]_1$ to get

$$[\tilde{W}]_2 = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}, \quad [\tilde{W}]_2^T \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} [\tilde{W}]_2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{matrix} u_2 \\ a_2 \end{matrix}$$

$$[\tilde{W}]_2^T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} [\tilde{W}]_2 = \begin{bmatrix} 1/4 & 1/2 \\ 1/2 & 1 \end{bmatrix} \begin{matrix} u_2 \\ a_2 \end{matrix}$$

Synthesized structure, vibration problem:

$$\left(\begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{2} + 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} - \omega^2 \begin{bmatrix} 1 & 1/2 & 0 \\ 1/2 & \frac{5}{4} + \frac{1}{2} & 1/2 \\ 0 & 1/2 & 1 \end{bmatrix} \right) \begin{Bmatrix} a_1 \\ u_2 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Multiply by 2, substitute $\mu = 1/\omega^2$, switch signs:

$$\left(\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} - \mu \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 8 \end{bmatrix} \right) \begin{Bmatrix} a_1 \\ u_2 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{vmatrix} 2-4\mu & 1 & 0 \\ 1 & 3-3\mu & 1 \\ 0 & 1 & 2-8\mu \end{vmatrix} = 0$$

By substituting $\mu_1 = \left(\frac{1}{0.9246}\right)^2$, $\mu_2 = \left(\frac{1}{1.574}\right)^2$, and $\mu_3 = \left(\frac{1}{2.381}\right)^2$,

we check that the determinant is indeed zero in each case.

11.10-1

Eq. 11.10-1, with $\xi = 0$: $\bar{u} = \frac{F_0/k}{\pm(1-\beta^2)}$, where $\beta = \frac{\Omega}{\omega}$

Want $\frac{\bar{u}}{F_0/k} = 1.10$

Positive root: $1.10 = \frac{1}{1-\beta^2}$, $\beta = 0.3015$

Negative root: $1.10 = -\frac{1}{1-\beta^2}$, $\beta = 1.3817$

$$0.3015 < \frac{\Omega}{\omega} < 1.3817$$

11.12-1

Forward: see Eq. 11.12-2a. Terms that contain Δt to second and higher powers discarded. This implies second-order accuracy according to the argument that follows Eq. 11.12-2b.

Backward: Write Eq. 11.12-2b for the next time step.

$$\underline{D}_n = \underline{D}_{n+1} - \Delta t \dot{\underline{D}}_{n+1} + \frac{\Delta t^2}{2} \ddot{\underline{D}}_{n+1} - \dots$$

Solve for $\underline{D}_{n+1}$ and discard Δt^2 and higher terms.

$$\underline{D}_{n+1} = \underline{D}_n + \Delta t \dot{\underline{D}}_{n+1}$$

Second-order accurate according to the same argument.

11.12-2

$$\underline{C} = 0 \text{ in Eq. 11.12-6: } \frac{1}{\Delta t^2} M_{\sim} D_{\sim n+1} = R_{\sim n}^{\text{ext}} - R_{\sim n}^{\text{int}} + \frac{1}{\Delta t^2} M_{\sim} (D_{\sim n} + \Delta t \dot{D}_{\sim n-\frac{1}{2}})$$

Write Eq. 11.2-5a for the previous time step: $D_{\sim n} = D_{\sim n-1} + \Delta t \dot{D}_{\sim n-\frac{1}{2}}$

Solve the latter equation for $\Delta t \dot{D}_{\sim n-\frac{1}{2}}$ and subs. into former equation:

$$\frac{1}{\Delta t^2} M_{\sim} D_{\sim n+1} = R_{\sim n}^{\text{ext}} - R_{\sim n}^{\text{int}} + \frac{1}{\Delta t^2} M_{\sim} (D_{\sim n} + D_{\sim n} - D_{\sim n-1})$$

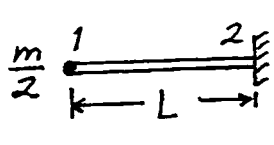
$\underbrace{D_{\sim n} + D_{\sim n} - D_{\sim n-1}}_{2D_{\sim n}}$

← agree

Eq. 11.12-3 with $\underline{C} = 0$ is

$$\frac{1}{\Delta t^2} M_{\sim} D_{\sim n+1} = R_{\sim n}^{\text{ext}} - R_{\sim n}^{\text{int}} + \frac{2}{\Delta t^2} M_{\sim} D_{\sim n} - \frac{1}{\Delta t^2} M_{\sim} D_{\sim n-1}$$

11.12-3


 Exact ω^2 is
 $\omega^2 = \frac{k}{m/2} = \frac{2AE}{mL}$

Element bound, from Eq. 11.12-15:

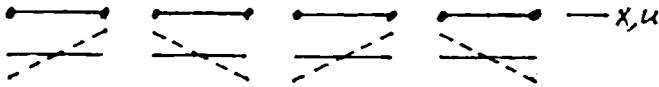
$$(\omega_{max})_e^2 = \frac{4E}{\rho L^2} = \frac{4EA}{L(\rho LA)} = \frac{4EA}{mL} \quad (100\% \text{ high})$$

Gershgorin bound: $[K] = \frac{AE}{L}$, $[M] = \frac{m}{2}$

$$\omega_{max}^2 \leq \frac{k}{m/2} \quad (\text{exact})$$

11.12-4

(a) Imagine identical but unconnected two-node bar elements, each unsupported, vibrating axially, and 180° out of phase with its neighbors on either side.



We can connect the elements without changing anything.

(b) If we fix one end of the model sketched in the solution of part (a), the vibration mode is perturbed there, but the disturbance is not much felt at the other end, particularly if there are a great many elements. Agreement improves.

11.12-5

$$\Delta t_{cr} = 2/\omega = 2.0$$

From Eq. 11.12-14, $u_1 = -\Delta t$

From Eq. 11.12-3

$$\frac{1}{\Delta t^2} u_{n+1} = -u_n + \frac{1}{\Delta t^2} (2u_n - u_{n-1})$$

$$u_{n+1} = (2 - \Delta t^2) u_n - u_{n-1}$$

(a) $\Delta t = 1$, $u_{n+1} = u_n - u_{n-1}$

n	-1	0	1	2	3	4	5
u_n	-1	0	1	1	0	-1	-1

(b) $\Delta t = \sqrt{2}$, $u_{n+1} = -u_{n-1}$

n	-1	0	1	2	3	4	5
u_n	$-\sqrt{2}$	0	$\sqrt{2}$	0	$-\sqrt{2}$	0	$\sqrt{2}$

(c) $\Delta t = 2$ (critical), $u_{n+1} = -2u_n - u_{n-1}$

n	-1	0	1	2	3	4	5
u_n	-2	0	2	-4	6	-8	10

Blows up in arithmetic fashion.

(d) $\Delta t = 3$ (>critical), $u_{n+1} = -7u_n - u_{n-1}$

n	-1	0	1	2	3	4	5
u_n	-3	0	3	-21	144	-1008	6912

Blows up in geometric fashion.

11.12-6

Note that $\Delta t_{cr} = \frac{2}{\omega} = \frac{2}{1} = 2.0$

Eq. 11.12-3: $\frac{1}{\Delta t^2} D_{n+1} = 1 - \left(1 - \frac{2}{\Delta t^2}\right) D_n - \frac{1}{\Delta t^2} D_{n-1}$

or $D_{n+1} = \Delta t^2 - (\Delta t^2 - 2) D_n - D_{n-1}$

Also $D_{-1} = 0 - 0 + \frac{\Delta t^2}{2} (1) = \frac{\Delta t^2}{2}$

(a) $\Delta t = 0.5$, $D_{-1} = 0.125$

$D_{n+1} = 0.250 + 1.750 D_n - D_{n-1}$

t	0	0.5	1.0	1.5	2.0
D	0	0.125	0.469	0.945	1.436

t	2.5	3.0	3.5	4.0	4.5
D	1.817	1.994	1.923	1.621	1.163

t	5.0	5.5	6.0	6.5	7.0
D	0.665	0.251	0.024	0.041	0.298

(b) $\Delta t = 1.0$, $D_{-1} = 0.50$

$D_{n+1} = 1.0 + D_n - D_{n-1}$

t	0	1	2	3	4	5	6	7
D	0	0.5	1.5	2.0	1.5	0.5	0	0.5

(c) $\Delta t = 2.0$, $D_{-1} = 2.0$

$D_{n+1} = 4 - 2D_n - D_{n-1}$

t	0	2	4	6	8	10
D	0	2	0	2	0	2

(d) $\Delta t = 3.0$, $D_{-1} = 4.5$

$D_{n+1} = 9 - 7D_n - D_{n-1}$

t	0	3	6	9	12	15
D	0	4.5	-22.5	162	-1103	7565

(e) $D + \ddot{D} = 1$, $D = A \sin t + B \cos t + 1$

At $t = 0$, $D = 0$, $\therefore B = -1$

At $t = 0$, $\dot{D} = 0$, $\therefore A = 0$ $D = 1 - \cos t$

t	0	0.5	1	1.5	2	2.5
D	0	0.122	0.460	0.929	1.416	1.801
t	3	3.5	4	4.5	5	5.5
D	1.990	1.937	1.654	1.211	0.716	0.291
t	6	6.5	7	8	9	10
D	0.040	0.023	0.246	1.146	1.911	1.839

11.12-7

Eq. 11.2-12, with proposed representation of viscous terms, is

$$\underline{\underline{M}} \ddot{\underline{\underline{D}}}_n + \alpha \underline{\underline{M}} \dot{\underline{\underline{D}}}_n + \beta \underline{\underline{K}} \dot{\underline{\underline{D}}}_{n-\frac{1}{2}} + \underline{\underline{K}} \underline{\underline{D}}_n = \underline{\underline{R}}_n^{\text{ext}}$$

Approximate $\dot{\underline{\underline{D}}}_n$ and $\ddot{\underline{\underline{D}}}_n$ by Eqs. 11.12-1. Leave $\dot{\underline{\underline{D}}}_{n-\frac{1}{2}}$ as-is so it can be computed with historical information. Thus

$$\frac{1}{\Delta t^2} \underline{\underline{M}} (\underline{\underline{D}}_{n+1} - 2\underline{\underline{D}}_n + \underline{\underline{D}}_{n-1}) + \frac{\alpha}{2\Delta t} \underline{\underline{M}} (\underline{\underline{D}}_{n+1} - \underline{\underline{D}}_{n-1}) + \beta \underline{\underline{K}} \dot{\underline{\underline{D}}}_{n-\frac{1}{2}} + \underline{\underline{K}} \underline{\underline{D}}_n = \underline{\underline{R}}_n^{\text{ext}}$$

Which is, after rearrangement,

$$\left(\frac{1}{\Delta t^2} + \frac{\alpha}{2\Delta t} \right) \underline{\underline{M}} \underline{\underline{D}}_{n+1} = \underline{\underline{R}}_n^{\text{ext}} - \underline{\underline{K}} \underline{\underline{D}}_n + \frac{1}{\Delta t^2} \underline{\underline{M}} \left[2\underline{\underline{D}}_n - \left(1 - \frac{\alpha \Delta t}{2} \right) \underline{\underline{D}}_{n-1} \right] - \beta \underline{\underline{K}} \dot{\underline{\underline{D}}}_{n-\frac{1}{2}}$$

Compute $\beta \underline{\underline{K}} \dot{\underline{\underline{D}}}_{n-\frac{1}{2}}$ in element-by-element fashion.

The contribution of one element to this is

$$\beta \underline{\underline{k}} \dot{\underline{\underline{d}}}_{n-\frac{1}{2}} = \beta \int \underline{\underline{B}}^T \underline{\underline{E}} \underline{\underline{B}} dV \dot{\underline{\underline{d}}}_{n-\frac{1}{2}} = \beta \int \underline{\underline{B}}^T \underline{\underline{E}} \dot{\underline{\underline{e}}}_{n-\frac{1}{2}} dV = \beta \int \underline{\underline{B}}^T \dot{\underline{\underline{\sigma}}}_{n-\frac{1}{2}} dV$$

We replace $\underline{\underline{K}} \underline{\underline{D}}_n$ by $\underline{\underline{R}}_n^{\text{int}} = \sum \underline{\underline{r}}_n^{\text{int}} = \sum \int \underline{\underline{B}}^T (\underline{\underline{\sigma}}_n + \beta \dot{\underline{\underline{\sigma}}}_{n-\frac{1}{2}}) dV$

11.13-1

$$D_{n+1} = D_n + \Delta t \dot{D}_n + \frac{\Delta t^2}{2} \ddot{D}_n + \frac{\Delta t^3}{6} \dddot{D}_n + \dots \quad (A)$$

$$D_n = D_{n+1} - \Delta t \dot{D}_{n+1} + \frac{\Delta t^2}{2} \ddot{D}_{n+1} - \frac{\Delta t^3}{6} \dddot{D}_{n+1} + \dots \quad (B)$$

Solve (B) for $\dot{D}_{n+1}$

$$D_{n+1} = D_n + \Delta t \dot{D}_{n+1} - \frac{\Delta t^2}{2} \ddot{D}_{n+1} + \frac{\Delta t^3}{6} \dddot{D}_{n+1} - \dots \quad (C)$$

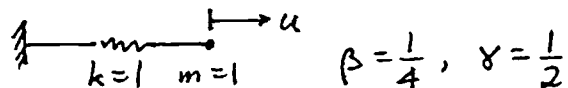
Add (A) & (C); divide result by 2.

$$D_{n+1} = D_n + \frac{\Delta t}{2} (\dot{D}_n + \dot{D}_{n+1}) + \frac{\Delta t^2}{4} (\ddot{D}_n - \ddot{D}_{n+1}) + \dots$$

(D) = result requested

Terms omitted from Eq. (D) depend on Δt^2 , so error is $O(\Delta t^2)$.

11.13-2



(a) $\Delta t = 2$

Eq. 11.13-5: $2u_{n+1} = u_n + 2\dot{u}_n + \ddot{u}_n$

Eq. 11.13-4b: $\dot{u}_{n+1} = u_{n+1} - u_n - \dot{u}_n$

Eq. 11.13-4a: $\ddot{u}_{n+1} = u_{n+1} - u_n - 2\dot{u}_n - \ddot{u}_n$

n	u_n	$\dot{u}_n$	$\ddot{u}_n$
0	0	1	0
1	1	0	-1
2	0	-1	0
3	-1	0	1
4	0	1	0

(b) $\Delta t = 1$

Eq. 11.13-5: $(4+1)u_{n+1} = 4(u_n + \dot{u}_n) + \ddot{u}_n$

or $u_{n+1} = 0.8(u_n + \dot{u}_n) + 0.2\ddot{u}_n$

Eq. 11.13-4b: $\dot{u}_{n+1} = 2(u_{n+1} - u_n) - \dot{u}_n$

Eq. 11.13-4a: $\ddot{u}_{n+1} = 4(u_{n+1} - u_n - \dot{u}_n) - \ddot{u}_n$

n	u_n	$\dot{u}_n$	$\ddot{u}_n$
0	0	1	0
1	0.80	0.6	-0.8
2	0.96	-0.28	-0.96
3	0.352	-0.963	-0.352
4	-0.5592		

11.14-1

$$(a) \quad 2\xi\omega \frac{z_{n+1} - z_n}{\Delta t} + \omega^2 z_n = 0$$

$$\text{or } z_{n+1} + \frac{\omega \Delta t}{\xi} z_n - z_n = 0 \quad (A)$$

$$\text{From Eq. 13.13-5: } z_n = C\lambda^n, z_{n+1} = C\lambda^{n+1}, z_{n-1} = C\lambda^{n-1}$$

So (A) becomes

$$C\lambda^{n+1} + \frac{\omega \Delta t}{\xi} C\lambda^n - C\lambda^{n-1} = 0$$

$$\text{Divide by } C\lambda^{n-1}; \lambda^2 + \frac{\omega \Delta t}{\xi} \lambda - 1 = 0$$

$$\lambda_{1,2} = \frac{1}{2} \left[-\frac{\omega \Delta t}{\xi} \pm \sqrt{\frac{\omega^2 \Delta t^2}{\xi^2} - 4(1)(-1)} \right]$$

↖ always real

Since $\lambda_1 \lambda_2 = c/a = -1$, & since λ_1 & λ_2 are real & distinct, then one $|\lambda| < 1$ while other $|\lambda| > 1$ and method is unstable.

$$(b) \quad 2\xi\omega \underbrace{(\dot{z}_{n+1} + \dot{z}_n)}_{\frac{2}{\Delta t}(z_{n+1} - z_n)} + \omega^2(z_{n+1} + z_n) = 0$$

$$z_{n+1} - z_n + \frac{\omega \Delta t}{4\xi}(z_{n+1} + z_n) = 0$$

$$\left(1 + \frac{\omega \Delta t}{4\xi}\right) z_{n+1} + \left(\frac{\omega \Delta t}{4\xi} - 1\right) z_n = 0. \text{ Subs. } z_n = C\lambda^n, z_{n+1} = C\lambda^{n+1}$$

$$\left(1 + \frac{\omega \Delta t}{4\xi}\right) \lambda + \frac{\omega \Delta t}{4\xi} - 1 = 0$$

$$\lambda = \left(1 - \frac{\omega \Delta t}{4\xi}\right) \frac{1}{1 + \frac{\omega \Delta t}{4\xi}} \quad \text{Let } h = \frac{\omega \Delta t}{4\xi}$$

↖ always positive

$$|\lambda| = \left| \frac{1-h}{1+h} \right| < 1 \text{ for all } h \text{ (i.e. for all } \Delta t \text{):}$$

unconditionally stable

$$(c) \quad 2\xi\omega \dot{z}_n + \omega^2 z_n = 0$$

$$2\xi\omega \frac{1}{\Delta t}(z_{n+1} - z_n) + \omega^2 z_n = 0$$

$$z_{n+1} + \left(\frac{\omega \Delta t}{2\xi} - 1\right) z_n = 0 \quad \text{Subs. } z_n = C\lambda^n, z_{n+1} = C\lambda^{n+1}$$

$$\lambda + \left(\frac{\omega \Delta t}{2\xi} - 1\right) = 0 \text{ or } \lambda = 1 - \frac{\omega \Delta t}{2\xi}$$

↖ always positive

For $|\lambda| \leq 1$, must have

$$-1 \leq 1 - \frac{\omega \Delta t}{2\xi} \leq 1 \text{ or } -2 \leq -\frac{\omega \Delta t}{2\xi} \leq 0$$

↖ always satisfied

Then $-2 \leq -\frac{\omega \Delta t}{2\xi}, \frac{\omega \Delta t}{2\xi} \leq 2$, i.e. must have

$$\Delta t \leq \frac{4\xi}{\omega} \text{ for stability (conditionally stable)}$$

$$(d) \quad 2\xi\omega \dot{z}_{n+1} + \omega^2 z_{n+1} = 0$$

$$2\xi\omega \frac{1}{\Delta t}(z_{n+1} - z_n) + \omega^2 z_{n+1} = 0$$

$$z_{n+1} - z_n + \frac{\omega \Delta t}{2\xi} z_{n+1} = 0 \quad \text{Subs. } z_n = C\lambda^n, \text{ etc.}$$

$$\left(1 + \frac{\omega \Delta t}{2\xi}\right) \lambda - 1 = 0, \quad \lambda = \frac{1}{1 + \frac{\omega \Delta t}{2\xi}}$$

↖ always pos.

$$|\lambda| < 1 \text{ for all } \Delta t \quad \text{unconditionally stable}$$

11.14-2

In Prob. 11.12-5, $k=1$ and $\omega=1$, so
 $\omega_{\text{exact}}=1$ and $P_{\text{exact}}=2\pi$.

(a) $\Delta t=1$, period $=6\Delta t=6$

$$P = \frac{2\pi/b}{2\pi/\omega} = \frac{6}{2\pi} = \frac{3}{\pi} = 0.9549$$

Eq. 11.14-20:

$$P = \omega \Delta t \left(\tan^{-1} \frac{(1)\sqrt{4-1}}{2-1} \right)^{-1} = (1) \frac{1}{\tan^{-1} \sqrt{3}} = 0.9549$$

(b) $\Delta t = \sqrt{2}$, period $= 4\Delta t = 4\sqrt{2}$

$$P = \frac{2\pi/b}{2\pi/\omega} = \frac{4\sqrt{2}}{2\pi} = \frac{2\sqrt{2}}{\pi}$$

Eq. 11.14-20:

$$P = \omega \Delta t \left(\tan^{-1} \frac{\sqrt{2}}{2-2} \right)^{-1} = \frac{\sqrt{2}}{\pi/2} = \frac{2\sqrt{2}}{\pi}$$

(c) $\Delta t = 2$, period $= 2\Delta t = 4$, $P = \frac{4}{2\pi} = \frac{2}{\pi}$

Eq. 11.14-20:

$$P = \omega \Delta t \left(\tan^{-1} \frac{0}{-2} \right)^{-1} = \frac{2}{\pi}$$

11.14-3

In Prob. 11.13-2, $k=1$ and $m=1$, so $\omega_{\text{exact}}=1$ and $P_{\text{exact}}=2\pi$.

$$(a) \quad \text{Period} = 4\Delta t = 4(2) = 8$$
$$P = \frac{2\pi/b}{2\pi/\omega} = \frac{8}{2\pi} = \frac{4}{\pi}$$

Eq. 11.14-21:

$$P = 2 \left(\tan^{-1} \frac{4(2)}{4-4} \right)^{-1} = \frac{2}{\pi/2} = \frac{4}{\pi}$$

(b) Determine when $u_n=0$ by a linear interpolation approximation.

$$n \approx 3 + \frac{0.352}{0.352+0.559} = 3.39$$

$$\text{period} \approx 3.39 \Delta t = 3.39$$

$$P = \frac{2\pi/b}{2\pi/\omega} = \frac{2(3.39)}{2\pi} = \frac{3.39}{\pi} = 1.08$$

Eq. 11.14-21:

$$P = (1) \left(\tan^{-1} \frac{4}{4-1} \right)^{-1} = \frac{1}{\tan \frac{4}{3}} = 1.08$$

11.14-4

(a) Eq. 11.14-20:

$$\omega \Delta t = 0: P = 0$$

$$\omega \Delta t = 1: P = (1) \left[\tan^{-1} \frac{\sqrt{3}}{1} \right]^{-1} = 0.955$$

$$\omega \Delta t = \sqrt{2}: P = \sqrt{2} \left[\tan^{-1} \frac{\sqrt{2}\sqrt{2}}{0} \right]^{-1} = \frac{\sqrt{2}}{\pi/2} = 0.900$$

$$\omega \Delta t = 2: P = 2 \left[\tan^{-1} \frac{2(0)}{-2} \right]^{-1} = \frac{2}{\pi} = 0.637$$

(b) Eq. 11.14-21: $\omega \Delta t = 0: P = 0$

$$\omega \Delta t = 1: P = (1) \left[\tan^{-1} \frac{4}{3} \right]^{-1} = \frac{1}{0.9273} = 1.078$$

$$\omega \Delta t = 2: P = (2) \left[\tan^{-1} \frac{8}{0} \right]^{-1} = \frac{2}{\pi/2} = \frac{4}{\pi} = 1.273$$

$$\omega \Delta t = 4: P = (4) \left[\tan^{-1} \frac{16}{-12} \right]^{-1} = \frac{4}{2.214} = 1.806$$

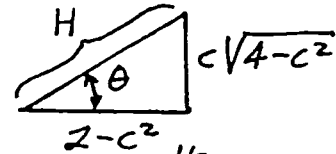
11.14-5

(a) Substitute $b \Delta t$ from Eq. 11.14-19 into Eq. 11.14-22:

$$2 - \omega^2 \Delta t^2 = 2 e^{a \Delta t} \cos \left[\arctan \frac{\omega \Delta t \sqrt{4 - \omega^2 \Delta t^2}}{2 - \omega^2 \Delta t^2} \right] \quad \text{Solve for } a \Delta t:$$

$$a \Delta t = \ln \left(\frac{1 - \omega^2 \Delta t^2 / 2}{\cos \left[\arctan \frac{\omega \Delta t \sqrt{4 - \omega^2 \Delta t^2}}{2 - \omega^2 \Delta t^2} \right]} \right) \quad \text{Let } c = \omega \Delta t;$$

$$a \Delta t = \ln \left(\frac{1 - (c^2/2)}{\cos \left[\arctan \frac{c \sqrt{4 - c^2}}{2 - c^2} \right]} \right)$$



$$H = \left[(2 - c^2)^2 + (c \sqrt{4 - c^2})^2 \right]^{1/2} = \left[4 - 4c^2 + c^4 + 4c^2 - c^4 \right]^{1/2} = 4^{1/2} = 2$$

$$\text{Hence } \cos \theta = \frac{2 - c^2}{2} = 1 - (c^2/2) \quad \text{and } a \Delta t = \ln 1 = 0$$

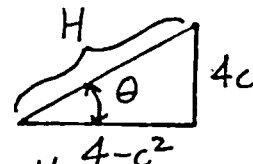
(b) Obtain λ_1, λ_2 from Eq. 11.14-13 and 11.14-16; thus

$$e^{a \Delta t} \cos b \Delta t = \frac{4 - \omega^2 \Delta t^2}{4 + \omega^2 \Delta t^2} \quad \text{Then from Eq. 11.14-21, where } h = \omega^2 \Delta t^2 / 4,$$

$$\tan b \Delta t = \frac{4 \omega \Delta t}{4 - \omega^2 \Delta t^2} \quad \text{Combine these two eqs. to obtain}$$

$$e^{a \Delta t} \cos \left[\arctan \frac{4 \omega \Delta t}{4 - \omega^2 \Delta t^2} \right] = \frac{4 - \omega^2 \Delta t^2}{4 + \omega^2 \Delta t^2} \quad \text{With } c = \omega \Delta t, \text{ we get}$$

$$a \Delta t = \ln \left(\frac{4 - c^2}{(4 + c^2) \cos \left[\arctan \frac{4c}{4 - c^2} \right]} \right)$$



$$H = \left[(4 - c^2)^2 + (4c)^2 \right]^{1/2} = \left[16 - 8c^2 + c^4 + 16c^2 \right]^{1/2} = 4 + c^2$$

$$\therefore \cos \theta = \frac{4 - c^2}{4 + c^2}, \text{ and}$$

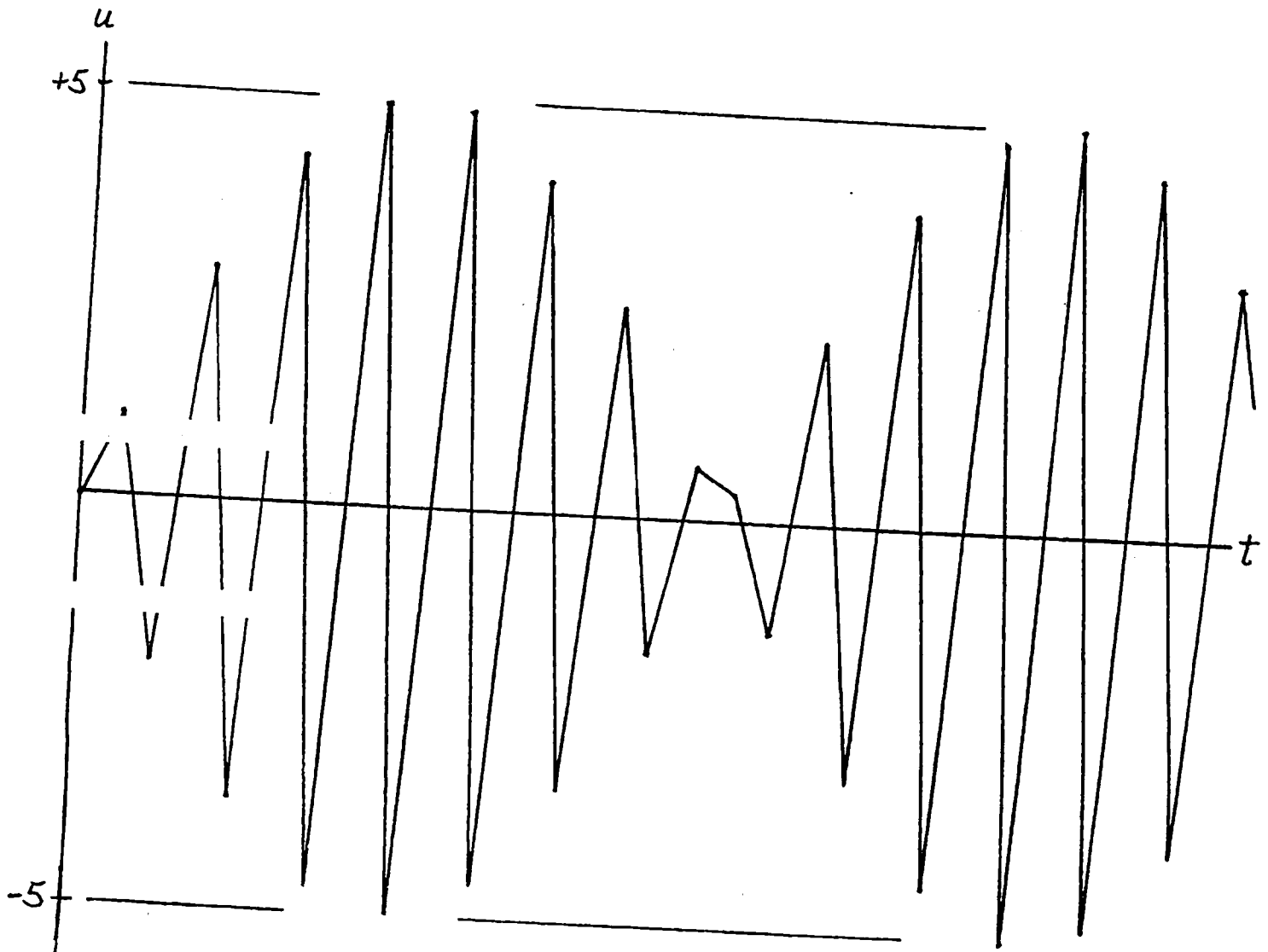
$$a \Delta t = \ln \frac{4 - c^2}{(4 + c^2) \frac{4 - c^2}{4 + c^2}} = \ln 1 = 0$$

11.14-6

Note that $\Delta t_{cr} = 2$, so we operate just under the stability limit.

$$u_{n+1} = (2 - \Delta t^2)u_n - u_{n-1} = -1.96u_n - u_{n-1}$$

n	u_n	n	u_n	n	u_n	n	u_n
-1	-1	9	4.89	19	-3.10	29	-2.29
0	0	10	-4.56	20	3.83	30	1.36
1	1	11	4.05	21	-4.40	31	-0.37
2	-1.96	12	-3.38	22	4.79	32	-0.64
3	2.84	13	2.57	23	-5.00	33	1.62
4	-3.61	14	-1.66	24	5.00	34	-2.53
5	4.23	15	0.68	25	-4.81	35	3.35
6	-4.69	16	0.32	26	4.42	36	-4.02
7	4.95	17	-1.32	27	-3.85	37	4.52
8	-5.02	18	2.25	28	3.14	38	-4.88



11.17-1

Node 51 amplitude, mode 3,
from Table 11.17-1, is 0.0034.

Modal load, mode 3:

$$P_3 = 0.0034(3000) = 10.2$$

From Fig. 11.17-2, $f_3 = 70.77 \text{ Hz}$

$$\text{hence } \omega_3 = 2\pi f_3 = 444.7 /s$$

With $\beta_3 = 1$, Eq. 11.10-1 yields ($\xi_3 = 0.02$)

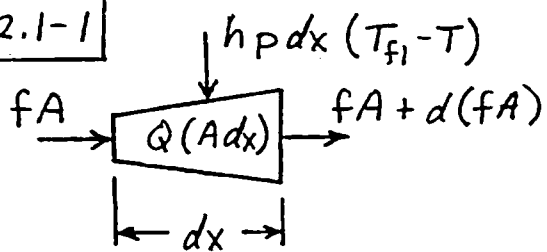
$$z_3 = \frac{P_3/\omega_3^2}{2\xi_3} = 1.28(10)^{-3}$$

Node 16 amplitude, mode 3, from
Table 11.17-1, is 0.0238. Hence

$$\bar{u}_{16} = 0.0238 z_3 = 30.7(10)^{-6} \text{ m}$$

which agrees with value plotted in
Fig. 11.17-3b.

12.1-1



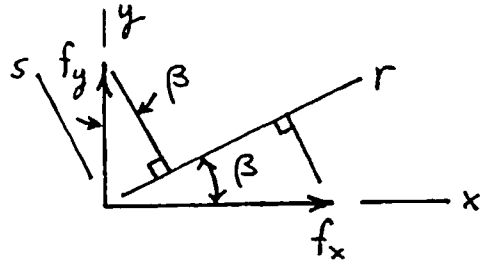
$$fA + h_p dx (T_{fi} - T) + Q A dx - [fA + d(fA)] = c \rho \dot{T} A dx$$

$$-\frac{d}{dx}(fA) + h_p (T_{fi} - T) + QA - c \rho \dot{T} A = 0$$

Substitute $f = -k T_x$

$$\frac{d}{dx}(A k T_x) + A Q + h_p (T_{fi} - T) - A c \rho \dot{T} = 0$$

12.1-2



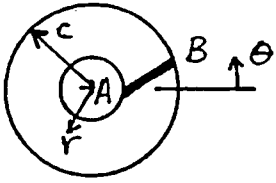
$$\begin{aligned} f_r &= f_x \cos \beta + f_y \sin \beta \\ f_s &= -f_x \sin \beta + f_y \cos \beta \end{aligned} \quad \text{or} \quad \begin{Bmatrix} f_r \\ f_s \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{bmatrix}}_{[\tilde{\Lambda}]} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix}$$

But $[\tilde{\Lambda}]$ is orthogonal; $[\tilde{\Lambda}]^{-1} = [\tilde{\Lambda}]^T = \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix}$

Therefore $\begin{Bmatrix} f_x \\ f_y \end{Bmatrix} = [\tilde{\Lambda}]^T \begin{Bmatrix} f_r \\ f_s \end{Bmatrix}$

12.1-3

Consider a disk, viewed along its axis. Let AB be a path that is much more conductive than the remainder. If A is heated, B will be much hotter than other parts of $r=c$. Thus, T on $r=c$ is not symmetric w.r.t. $\theta=0$, but would become so if AB is radial.



12.1-4

(a) Use $\{\underline{\partial}\}$ and $\{\underline{T}_{\partial}\}$ from Eqs. 12.1-15

$$[\underline{K}]\{\underline{T}_{\partial}\} = k \left[T_{,r} \quad \frac{1}{r} T_{,\theta} \quad T_{,z} \right]^T$$

Eq. 12.1-14a becomes

$$k \left(\frac{1}{r} T_{,r} + T_{,rr} + \frac{1}{r^2} T_{,\theta\theta} + T_{,zz} \right) + Q - c \rho \dot{T} = 0$$

Eq. 12.1-14b becomes

$$f_B = k (\ell T_{,r} + n T_{,z})$$

$$(b) \{\underline{\partial}\} = \left\{ \frac{1}{r} + \frac{\partial}{\partial r} \right\}, \{\underline{T}_{\partial}\} = \left\{ T_{,r} \right\}$$

Eq. 12.1-14a:

$$\{\underline{\partial}\}^T \left\{ \begin{matrix} K_{11} T_{,r} + K_{12} \frac{1}{r} T_{,\theta} \\ K_{21} T_{,r} + K_{22} \frac{1}{r} T_{,\theta} \end{matrix} \right\} + Q = c \rho \dot{T} \quad (K_{12} = K_{21})$$

$$K_{11} T_{,rr} + K_{11} \frac{1}{r} T_{,r} + K_{12} \frac{1}{r^2} T_{,\theta} - K_{12} \frac{1}{r^2} T_{,\theta} + K_{12} \frac{1}{r} T_{,r\theta}$$

$$+ K_{21} \frac{1}{r} T_{,r\theta} + K_{22} \frac{1}{r^2} T_{,\theta\theta} + Q = \rho c \dot{T}. \text{ Gather terms.}$$

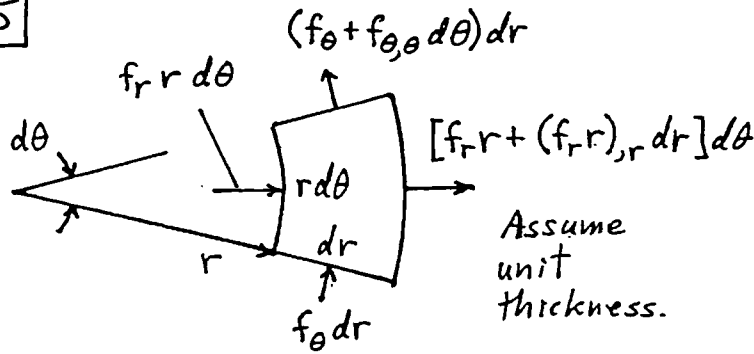
$$K_{11} \left(T_{,rr} + \frac{1}{r} T_{,r} \right) + 2K_{12} \frac{1}{r} T_{,r\theta} + K_{22} \frac{1}{r^2} T_{,\theta\theta} + Q = c \rho \dot{T}$$

In Eq. 12.1-14b, $\ell_B = 1$ & $m_B = 0$.

$$f_B = \begin{bmatrix} 1 & 0 \end{bmatrix} \left\{ \begin{matrix} K_{11} T_{,r} + K_{12} \frac{1}{r} T_{,\theta} \\ K_{21} T_{,r} + K_{22} \frac{1}{r} T_{,\theta} \end{matrix} \right\}$$

$$f_B = K_{11} T_{,r} + K_{12} \frac{1}{r} T_{,\theta}$$

12.1-5



Net inward flux from the above is

$$-[f_r r + (f_r r)_{,r} dr] d\theta + f_r r d\theta - (f_\theta + f_{\theta,\theta} d\theta) dr + f_\theta dr$$

or $-[(f_r r)_{,r} + f_{\theta,\theta}] dr d\theta$

Net inward heat flow per unit volume is

$$-(r f_{r,r} + f_r + f_{\theta,\theta}) dr d\theta + Q r dr d\theta$$

Set equal to $\rho c \dot{T} r dr d\theta$ and divide by $r dr d\theta$

$$-f_{r,r} - \frac{1}{r} f_r - \frac{1}{r} f_{\theta,\theta} + Q = \rho c \dot{T} \quad (A)$$

If orthotropic, $f_r = -K_{11} T_{,r} - K_{12} \frac{1}{r} T_{,\theta}$

$$f_\theta = -K_{21} T_{,r} - K_{22} \frac{1}{r} T_{,\theta}$$

$$(K_{12} = K_{21})$$

Eq. (A) becomes

$$K_{11} T_{,rr} - K_{12} \frac{1}{r^2} T_{,\theta} + K_{12} \frac{1}{r} T_{,r\theta} + K_{11} \frac{1}{r} T_{,r} + K_{12} \frac{1}{r^2} T_{,\theta} + K_{21} \frac{1}{r} T_{,r\theta} + K_{22} \frac{1}{r^2} T_{,\theta\theta} + Q = \rho c \dot{T}$$

Gather terms

$$K_{11} (T_{,rr} + \frac{1}{r} T_{,r}) + 2K_{12} \frac{1}{r} T_{,r\theta} + K_{22} \frac{1}{r^2} T_{,\theta\theta} + Q - \rho c \dot{T} = 0$$

12.2-1

$$\text{Eq. 4.7-6 is } \frac{\partial F}{\partial T} - \frac{\partial}{\partial x} \frac{\partial F}{\partial T_x} - \frac{\partial}{\partial y} \frac{\partial F}{\partial T_y} = 0 \quad (A)$$

$$\text{where, from Eq. 12.2-1, } F = \frac{1}{2} \{T_\partial\}^T [\kappa] \{T_\partial\} - QT + c\rho \dot{T} T \quad (B)$$

$$\text{in which, for a plane problem, } \{T_\partial\} = \begin{Bmatrix} T_x \\ T_y \end{Bmatrix} \quad (C)$$

Rewriting (A), with $\{\partial\} = \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \end{Bmatrix}$,

$$\frac{\partial F}{\partial T} - \{\partial\}^T \begin{Bmatrix} \partial F / \partial T_x \\ \partial F / \partial T_y \end{Bmatrix} = 0 \quad (D)$$

Substitute (B) into (D)

$$-Q + c\rho \dot{T} - \{\partial\}^T ([\kappa] \{T_\partial\}) = 0$$

12.2-2

$$\begin{aligned} \Pi = & \iint \left(\frac{1}{2} k_x T_{,x}^2 + k_{xy} T_{,x} T_{,y} + \frac{1}{2} k_y T_{,y}^2 \right. \\ & \left. - Q T - 2 h T_f T + h T^2 + \rho c \dot{T} T \right) dx dy \\ & - \int h (T_f T - \frac{1}{2} T^2) dS - \int f_B T dS \\ \delta \Pi = 0 = & \iint \left(k T_{,x} \delta T_{,x} + k_{xy} T_{,x} \delta T_{,y} + k_{xy} T_{,y} \delta T_{,x} \right. \\ & \left. + k_y T_{,y} \delta T_{,y} - Q \delta T - 2 h T_f \delta T + 2 h T \delta T \right. \\ & \left. + \rho c \dot{T} \delta T \right) dx dy - \int h (T_f \delta T - T \delta T) dS - \int f_B \delta T dS \end{aligned}$$

Integrations by parts:

$$\iint k_x T_{,x} \delta T_{,x} dx dy = - \iint (k_x T_{,x})_{,x} \delta T dx dy + \int k_x T_{,x} \delta T l_B dS$$

$$\iint k_{xy} T_{,x} \delta T_{,y} dx dy = - \iint (k_{xy} T_{,x})_{,y} \delta T dx dy + \int k_{xy} T_{,x} \delta T m_B dS$$

Similar for next 2 terms.

$$\begin{aligned} \text{Thus} \\ \delta \Pi = 0 = & \iint \left[-(k_x T_{,x} + k_{xy} T_{,y})_{,x} - (k_{xy} T_{,x} + k_y T_{,y})_{,y} \right. \\ & \left. - Q - 2 h (T_f - T) + \rho c \dot{T} \right] \delta T dx dy + \int \left[(k_x T_{,x} + k_{xy} T_{,y}) l_B \right. \\ & \left. + (k_{xy} T_{,x} + k_y T_{,y}) m_B - h (T_f - T) - f_B \right] \delta T dS \end{aligned}$$

Vanishing of [---] in double integral yields Eq. 12.1-7. Vanishing of [---] in surface integral yields Eq. 12.1-12b. Lateral surface added to both.

12.2-3

Assume $\tilde{T} = \tilde{N} T_e$; use Eq. 12.1-10

$$\int_0^L \tilde{N}^T \left[(Ak \tilde{T}_{,x})_{,x} + QA + h(T_f - \tilde{T})_p \right] dx = 0$$

Integrate by parts:

$$\int_0^L \tilde{N}^T \left[(Ak \tilde{T}_{,x})_{,x} \right] dx = - \int_0^L \tilde{N}_{,x}^T Ak \tilde{T}_{,x} dx + \left(\tilde{N}^T Ak \tilde{T}_{,x} \right)_0^L$$

$$\text{But } \left(\tilde{N}^T Ak \tilde{T}_{,x} \right)_0^L = - \left(\tilde{N}^T Af \right)_0^L = \begin{Bmatrix} Af_0 \\ -Af_L \end{Bmatrix}$$

Also subs. $\tilde{T} = \tilde{N} T_e$ & $\tilde{T}_{,x} = \tilde{N}_{,x} T_e$ into the first (residual) equation. Thus

$$\underbrace{- \int_0^L \tilde{N}_{,x}^T Ak \tilde{N}_{,x} dx}_{[k]} T_e + \underbrace{\int_0^L \tilde{N}^T Q A dx}_{\{r_a\}} - \underbrace{\int_0^L \tilde{N}^T h_p \tilde{N} dx}_{[h_{ls}]} T_e + \underbrace{\int_0^L \tilde{N}^T h_p T_f dx}_{\{r_{ls}\}} + \begin{Bmatrix} Af_0 \\ -Af_L \end{Bmatrix} = 0$$

$$([k] + [h_{ls}]) \{T_e\} = \{r_a\} + \{r_{ls}\} + \begin{Bmatrix} Af_0 \\ -Af_L \end{Bmatrix}$$

Say Af is positive when directed into el.; thus $Af_0 = Af_1$ & $-Af_L = Af_2$

$$([k] + [h_{ls}]) \{T_e\} = \{r_a\} + \{r_{ls}\} + \begin{Bmatrix} Af_1 \\ Af_2 \end{Bmatrix}$$

12.2-4

$$(a) \begin{Bmatrix} T_x \\ T_y \end{Bmatrix} = [\underline{B}] \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} \text{ where, from Eqs.}$$

7.2-4 and 7.2-6,

$$[\underline{B}] = \frac{1}{2A} \begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix}, 2A = x_{21}y_{31} - x_{31}y_{21}$$

$$[\underline{k}] = k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ so } [\underline{k}] = Ak [\underline{B}]^T [\underline{B}]$$

$$(b) \text{ On side 1-3, } \xi_2 = 0, \text{ \& } [\underline{N}] = [\xi_1, 0, \xi_3]$$

Use Eq. 7.3-5 to integrate along 1-3.

$$[\underline{h}] = h \int \begin{bmatrix} \xi_1^2 & 0 & \xi_1 \xi_3 \\ 0 & 0 & 0 \\ \xi_1 \xi_3 & 0 & \xi_3^2 \end{bmatrix} dL_{13} = \frac{h L_{13}}{6} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$\text{where } L_{13} = [(x_3 - x_1)^2 + (y_3 - y_1)^2]^{1/2}.$$

$$(c) [\underline{h}] = \frac{h L_{13}}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, [\underline{c}] = \frac{\rho c A}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(d) [\underline{N}] = [\xi_1, \xi_2, \xi_3]. \text{ By Eq. 7.3-7,}$$

$$\int_A \xi_i dA = 2A \frac{1}{3!} = \frac{A}{3} \text{ for } i=1, 2, 3$$

Hence, with Q constant, & unit thickness,

$$\{\underline{r}_Q\} = Q \int [\underline{N}]^T dA = \frac{QA}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

12.2-5

(a) As in Prob. 12.2-4,

$$[\underline{B}] = \frac{1}{2A} \begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix}, \quad 2A = x_{21}y_{31} - x_{31}y_{21}$$

$$\text{Then } [\underline{k}] = \int [\underline{B}]^T [\underline{B}] k dV = k [\underline{B}]^T [\underline{B}] \int 2\pi r dA$$

$$r = [\xi_1 \ \xi_2 \ \xi_3] \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix} \text{ and } \int \xi_i dA = \frac{A}{3} \text{ for}$$

$i=1, 2, 3$ from Eq. 7.3-7. Thus

$$\int 2\pi r dA = 2\pi \frac{A}{3} (r_1 + r_2 + r_3)$$

(b) Integral for $[\underline{h}]$ in Prob. 12.2-4, applies, except dL_{13} becomes $2\pi r dL_{13}$ where $r = \xi_1 r_1 + \xi_3 r_3$ along side 1-3.

By Eq. 7.3-5,

$$\int \xi_1^3 dL_{13} = \int \xi_3^3 dL_{13} = \frac{L_{13}}{4}, \quad \int \xi_1^2 \xi_3 dL_{13} = \int \xi_1 \xi_3^2 dL_{13} = \frac{L_{13}}{12}$$

Hence

$$[\underline{h}] = h(2\pi L_{13}) \begin{bmatrix} \frac{r_1}{4} + \frac{r_3}{12}, 0, \frac{r_1+r_3}{12} \\ 0 & 0 & 0 \\ \frac{r_1+r_3}{12}, 0, \frac{r_1}{12} + \frac{r_3}{4} \end{bmatrix}$$

(c)

$$[\underline{h}] = 2\pi \frac{r_1+r_3}{2} L_{12} h \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\underline{c}] = 2\pi \frac{r_1+r_2+r_3}{3} \frac{\rho c A}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(d)

$$\{\underline{x}_Q\} = Q \left\{ \begin{Bmatrix} \xi_1 \\ \xi_2 \end{Bmatrix} \right\} 2\pi \underbrace{(\xi_1 r_1 + \xi_2 r_2 + \xi_3 r_3)}_r dA$$

$$\text{By Eq. 7.3-7, } \int \xi_i^2 dA = \frac{A}{6} \quad \& \quad \int \xi_i \xi_j dA = \frac{A}{12}$$

$$\{\underline{x}_Q\} = 2\pi Q \frac{A}{12} \begin{Bmatrix} 2r_1 + r_2 + r_3 \\ r_1 + 2r_2 + r_3 \\ r_1 + r_2 + 2r_3 \end{Bmatrix}$$

12.2-6

$$T = \underbrace{\begin{bmatrix} r_2 - r & r - r_1 \\ r_2 - r_1 & r_2 - r_1 \end{bmatrix}}_{[N]} \underbrace{\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}}_{\{T_h\}}, T_{,r} = \frac{1}{r_2 - r_1} \underbrace{\begin{bmatrix} -1 & 1 \end{bmatrix}}_{[B]} \underbrace{\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}}_{\{T_h\}}$$

$$[k] = \int_{-\pi}^{\pi} \int_{r_1}^{r_2} \underbrace{[B]^T}_{[N]} \underbrace{[B]}_{[B]} k r dr d\theta = 2\pi \frac{r_2^2 - r_1^2}{2(r_2 - r_1)^2} k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k] = \frac{r_2 + r_1}{r_2 - r_1} \pi k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$[h]$ and $\{r_h\}$ are not present on lateral surfaces

For possible convection on edge $r=r_1$,

$$[h] = \int_{-\pi}^{\pi} h \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} (1) r_1 d\theta = 2\pi h r_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\{r_h\} = \int_{-\pi}^{\pi} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} h T_{f1} (1) r_1 d\theta = 2\pi h T_{f1} r_1 \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

or, for convection on edge $r=r_2$,

$$[h] = 2\pi h r_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \{r_h\} = 2\pi h T_{f1} r_2 \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$\{r_B\}$ replace $h T_{f1}$ by f_B in $\{r_h\}$ expressions

$$[\varepsilon] = \int_{-\pi}^{\pi} \int_{r_1}^{r_2} [N]^T [N] \rho c r dr d\theta$$

(algebraic expressions do not simplify)

$$[k_Q] = \frac{Q}{r_2 - r_1} \int_{r_1}^{r_2} \begin{Bmatrix} r_2 - r \\ r - r_1 \end{Bmatrix} r dr = \frac{2\pi Q}{r_2 - r_1} \begin{Bmatrix} r_2 \frac{r^2}{2} - \frac{r^3}{3} \\ \frac{r^3}{3} - r_1 \frac{r^2}{2} \end{Bmatrix}_{r_1}^{r_2}$$

$$\{r_Q\} = \frac{\pi Q}{r_2 - r_1} \begin{Bmatrix} r_2^3 - 3r_1^2 r_2 + 2r_1^3 \\ 2r_1^3 - 3r_1 r_2^2 + r_1^3 \end{Bmatrix}$$

12.2-7

$$\frac{Ak}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

Only relative temperatures matter in this problem, so impose $T_3 = 0$ (we will add T_3 to T_1 & T_2 after solving); also set $q_2 = 0$.

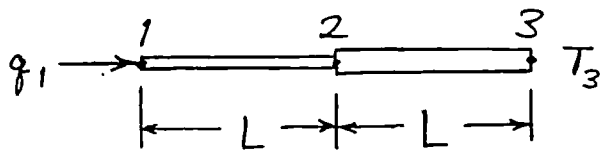
$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \frac{Lq_1}{Ak} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \frac{Lq_1}{Ak} \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$$

Re-introduce T_3 for final temps.

$$\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} T_3 \\ T_3 \end{Bmatrix} + \frac{Lq_1}{Ak} \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$$

$$q_3 = A \left[-k \frac{T_3 - T_2}{L} \right] = -\frac{Ak}{L} \left(-\frac{Lq_1}{Ak} \right) = q_1 \quad \checkmark$$

12.2-8



$$\frac{A_0 k}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+2 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

As in Problem 8.3, use $q_2 = 0$ & $T_3 = 0$.

$$\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \frac{L q_1}{A_0 k} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solve for T_1 and T_2 ; reintroduce T_3 .

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} T_3 \\ T_3 \end{bmatrix} + \frac{L q_1}{2 A_0 k} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$q_3 = 2 A_0 \left[-k \frac{T_3 - T_2}{L} \right] = -\frac{2 A_0 k}{L} \left(-\frac{L q_1}{2 A_0 k} \right) = q_1 \checkmark$$

12.2-9

$$\frac{Ak}{L} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1+1 & -1 & 0 \\ 0 & -1 & 1+1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

Impose $q_2 = q_3 = 0$, $T_1 = 0$, $T_4 = 300$
(the latter adds $300 Ak/L$ to the r.h.s.)

$$\frac{Ak}{L} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \frac{Ak}{L} \begin{Bmatrix} 0 \\ 300 \end{Bmatrix}$$

$$\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} 0 \\ 300 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 200 \end{Bmatrix}$$

(as expected)

12.2-10

$$\frac{A_0 k}{L} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1+2 & -2 & 0 \\ 0 & -2 & 2+3 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

Impose $q_2 = q_3 = 0$, $T_1 = 0$, $T_4 = 300$
(the latter adds $300(3A_0 k/L)$ to the

r.h.s.) $\frac{A_0 k}{L} \begin{bmatrix} 3 & -2 \\ -2 & 5 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \frac{A_0 k}{L} \begin{Bmatrix} 0 \\ 900 \end{Bmatrix}$

$$\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \frac{1}{11} \begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix} \begin{Bmatrix} 0 \\ 900 \end{Bmatrix} = \begin{Bmatrix} 163.6 \\ 245.5 \end{Bmatrix}$$

12.2-11

(a) With $\tilde{N} = \begin{bmatrix} \frac{L-x}{L} & \frac{x}{L} \end{bmatrix}$, $dS = p dx$
 where p = perimeter of cross section,

$$\int_0^L \tilde{N}^T \tilde{N} h dS = p h \int_0^L \begin{bmatrix} \left(\frac{L-x}{L}\right)^2 & \frac{L-x}{L} \frac{x}{L} \\ \frac{L-x}{L} \frac{x}{L} & \frac{x^2}{L^2} \end{bmatrix} dx$$

$$= \frac{p h L}{3} \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} = \frac{h S_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\int_0^L \tilde{N}^T h T_f dS = \int_0^L \begin{bmatrix} \frac{L-x}{L} \\ \frac{x}{L} \end{bmatrix} h T_f p dx = \frac{h T_f S_e}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

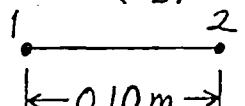
(b) $S_e = 0.010$

$$\left. \begin{aligned} \frac{h S_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} &= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\ \frac{A k}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} &= \begin{bmatrix} 0.4 & -0.4 \\ -0.4 & 0.4 \end{bmatrix} \end{aligned} \right\} \text{add: } \begin{bmatrix} 2.4 & 0.6 \\ 0.6 & 2.4 \end{bmatrix}$$

Also $\frac{h T_f S_e}{2} = 600$.

Combine elements and set $T_1 = 0$.

$$\begin{bmatrix} 4.8 & 0.6 \\ 0.6 & 2.4 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = 600 \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}, \quad \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 226 \\ 194 \end{Bmatrix}$$

(c)  $S_e = \frac{1}{3}(0.020)$ for the left element

$$\frac{H k}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 0.6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{Half this for the right el.}$$

$$\frac{h S_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 0.667 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{Twice this for the other el.}$$

$$\frac{h T_f S_e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = 400 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \text{Twice this for the other el.}$$

Assemble and set $T_1 = 0$.

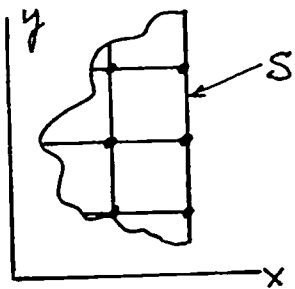
$$\begin{bmatrix} 0.6 + 0.3 + 2(0.667) & 1.333 - 0.3 \\ 1.333 - 0.3 & 2(1.333) + 0.3 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 1200 \\ 800 \end{Bmatrix}$$

$$\begin{bmatrix} 4.900 & 1.033 \\ 1.033 & 2.967 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 1200 \\ 800 \end{Bmatrix}$$

$$\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 203 \\ 199 \end{Bmatrix}$$

(d) The model change from (b) to (c) has made T_2 and T_3 closer to ambient and decreased the gradient from 2 to 3. The actual gradient is so close to node 1 that these coarse models cannot represent it (analytical equations for fins show as much).

12.2-12



Eq. 12.1-2:
$$\begin{Bmatrix} f_x \\ f_y \end{Bmatrix} = k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} T_x \\ T_y \end{Bmatrix}$$

	<u>Known</u>	<u>Unknown</u>
(a)	$T_x = 0$	$T \quad T_y$
(b)	$T \quad T_y$	T_x
(c)	T_x	$T \quad T_y$
(d)	————	$T \quad T_x \quad T_y$

12.3-1

$$\frac{Ak}{L} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} - \\ 0 \\ 0 \\ - \end{bmatrix}$$

Impose $T_1 = 0$, $T_4 = 500$, let $k = 73$.

$$73 \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 73(500) \end{bmatrix} \quad \text{from which}$$

$$T_2 = 166.7^\circ\text{C}$$

$$T_3 = 333.3^\circ\text{C}$$

In all elements,

$$f = -73 \frac{500 - 0}{3(0.04)} = -304,200 \frac{\text{W}}{\text{m}^2}$$

Compute el. ave. T 's & associated k 's.

el. 1, $T_{\text{ave}} = 83.3$, $k = 73 - 5 = 68$

el. 2, $T_{\text{ave}} = 250$, $k = 73 - 15 = 58$

el. 3, $T_{\text{ave}} = 416.7$, $k = 73 - 25 = 48$

$$\frac{A}{L} \begin{bmatrix} 68 & -68 & 0 & 0 \\ -68 & 68+58 & -58 & 0 \\ 0 & -58 & 58+48 & -48 \\ 0 & 0 & -48 & 48 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} - \\ 0 \\ 0 \\ - \end{bmatrix}$$

Impose $T_1 = 0$, $T_4 = 500$

$$\begin{bmatrix} 126 & -58 \\ -58 & 106 \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 48(500) \end{bmatrix} \quad \text{from which}$$

$$T_2 = 139.3^\circ\text{C}$$

$$T_3 = 302.6^\circ\text{C}$$

Element fluxes are

$$f_1 = -68 \frac{139.3 - 0}{0.04} = -236,800 \frac{\text{W}}{\text{m}^2}$$

$$f_2 = -58 \frac{302.6 - 139.3}{0.04} = -236,800 \frac{\text{W}}{\text{m}^2}$$

$$f_3 = -48 \frac{500 - 302.6}{0.04} = -236,800 \frac{\text{W}}{\text{m}^2}$$

The fluxes agree (a check).

Compute el. ave. T 's & associated k 's.

el. 1, $T_{\text{ave}} = 69.65$, $k = 68.8$

el. 2, $T_{\text{ave}} = 221.0$, $k = 59.7$

el. 3, $T_{\text{ave}} = 401.3$, $k = 48.9$

$$\frac{A}{L} \begin{bmatrix} 68.8 & -68.8 & 0 & 0 \\ -68.8 & 68.8+59.7 & -59.7 & 0 \\ 0 & -59.7 & 59.7+48.9 & -48.9 \\ 0 & 0 & -48.9 & 48.9 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} - \\ 0 \\ 0 \\ - \end{bmatrix}$$

Impose $T_1 = 0$, $T_4 = 500$.

$$\begin{bmatrix} 128.5 & -59.7 \\ -59.7 & 108.6 \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 48.9(500) \end{bmatrix}$$

from which $T_2 = 140.5^\circ\text{C}$, $T_3 = 302.4^\circ\text{C}$

Element fluxes are

$$f_1 = -68.8 \frac{140.5 - 0}{0.04} = -241,600 \frac{\text{W}}{\text{m}^2}$$

$$f_2 = -59.7 \frac{302.4 - 140.5}{0.04} = -241,600 \frac{\text{W}}{\text{m}^2}$$

$$f_3 = -48.9 \frac{500 - 302.4}{0.04} = -241,600 \frac{\text{W}}{\text{m}^2}$$

The fluxes agree (a check).

12.3-2

$$q = \frac{k}{t}(T_b - T_a) = h_r(T_c - T_b)$$

$$\text{hence } \left(\frac{k}{t} + h_r\right)T_b = \frac{k}{t}T_a + h_rT_c \quad (a)$$

$$\text{where } \frac{k}{t} = \frac{0.7}{0.01} = 70 \quad \text{and, from Eqs.}$$

8.3-2 and 8.3-4,

$$h_r = \frac{\sigma}{\frac{1}{0.6} + \frac{1}{0.6} - 1} (T_c^2 + T_b^2)(T_c + T_b)$$

Using absolute temperatures, $T_a = 293^\circ\text{K}$
and $T_c = 873^\circ\text{K}$, h_r & Eq. (a) become

$$h_r = 2.43(10)^{-8} (873^2 + T_b^2)(873 + T_b) \quad \&$$

$$(70 + h_r)T_b = 20,510 + 873h_r \quad (b)$$

First cycle: $T_b = 100^\circ\text{C} = 373^\circ\text{K}$; use (b):

$$(70 + 27.3)T_b = 20,510 + 873(27.3)$$

$$T_b = 455.7^\circ\text{K} \quad \text{Use this in (b);}$$

second cycle:

$$(70 + 31.3)T_b = 20,510 + 873(31.3)$$

$$T_b = 472.2^\circ\text{K} \quad \text{Use this in (c);}$$

third cycle:

$$(70 + 32.2)T_b = 20,510 + 873(32.2)$$

$$T_b = 475.8^\circ\text{K} = 202.8^\circ\text{C}$$

(close enough)

12.4-1

All symbols but λ are arrays

Given $\bar{T}_i^T C \bar{T}_i = 1$ (A)

Prove $\bar{T}_i^T C \bar{T}_j = 0$ for $i \neq j$ (B)

Premultiply Eq. 12.4-2 by $\bar{T}^T$:

(a) $\bar{T}_i^T K_T \bar{T}_j = \lambda_j \bar{T}_i^T C \bar{T}_j$ prem. jth by $\bar{T}_i^T$

(b) $\bar{T}_i^T K_T \bar{T}_i = \lambda_i \bar{T}_i^T C \bar{T}_i$ prem. ith by $\bar{T}_i^T$

(c) $\bar{T}_j^T K_T \bar{T}_j = \lambda_j \bar{T}_j^T C \bar{T}_j$ transpose of (b)

$0 = (\lambda_j - \lambda_i) \bar{T}_i^T C \bar{T}_j$ (a) - (c)

But $\lambda_i \neq \lambda_j$, so $\bar{T}_i^T C \bar{T}_j = 0$ Q.E.D.

If we repeat foregoing argument with Eq. 12.4-2 in form $(C - \frac{1}{\lambda} K_T) \bar{T} = 0$, get $\bar{T}_i^T K_T \bar{T}_j = 0$

To show second of Eqs. 12.4-3: we know

$\phi^T C \phi = I$ from (A) & (B). Hence

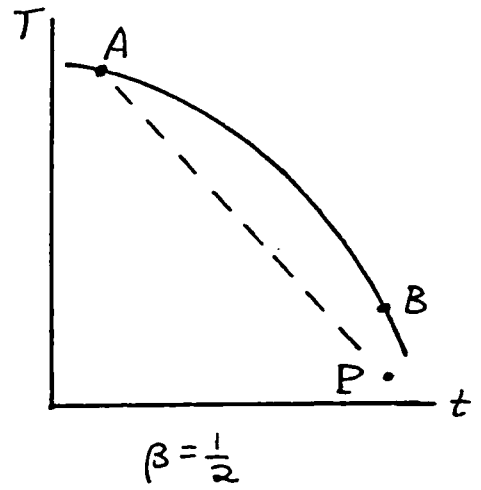
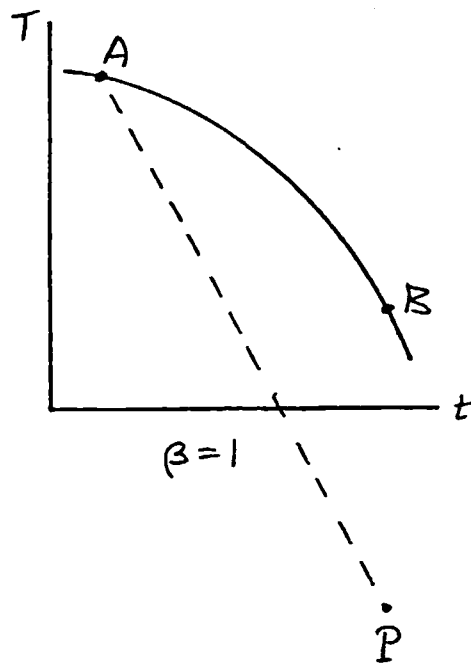
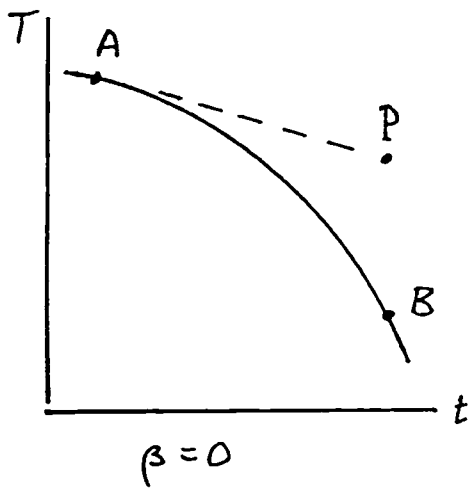
$\bar{T}_i^T (K_T - \lambda_j C) \bar{T}_j = 0$

$\bar{T}_i^T K_T \bar{T}_j = \lambda_j \bar{T}_i^T C \bar{T}_j \quad \begin{cases} = 0 \text{ for } i \neq j \\ = \lambda_j \text{ for } i = j \end{cases}$

Hence $\phi^T K_T \phi = [\lambda]$

12.4-2

P is the predicted T



12.4-3

(a) When $\dot{T} = 0$, $T = 3/6 = 0.500$

(b) Assume solution of form $T = a(1 - e^{-bt})$
Must have $a = 0.5$ to check part (a).

Thus $\dot{T} = 0.5be^{-bt}$ and diff. eq. becomes

$3(1 - e^{-bt}) + 1.0be^{-bt} = 3$. Therefore $b = 3$,
and exact solution is $T = 0.5(1 - e^{-3t})$

(c) $(6 - 2\lambda)\bar{T} = 0$, $\lambda = 3$, $\Delta t_{cr} = \frac{2}{\lambda} = \frac{2}{3}$

(d)-(k): Eq. 12.4-8 becomes

$$\left(\frac{1}{\Delta t} 2 + 6\beta\right) T_{n+1} = \left(\frac{1}{\Delta t} 2 - 6(1-\beta)\right) T_n + 3$$

(d)-(g): $\Delta t = 0.1$, $(20 + 6\beta)T_{n+1} = (14 + 6\beta)T_n + 3$

(h)-(k): $\Delta t = 1.0$, $(2 + 6\beta)T_{n+1} = (-4 + 6\beta)T_n + 3$

(d) $20T_{n+1} = 14T_n + 3$ (h) $2T_{n+1} = -4T_n + 3$

(e) $23T_{n+1} = 17T_n + 3$ (i) $5T_{n+1} = -T_n + 3$

(f) $24T_{n+1} = 18T_n + 3$ (j) $6T_{n+1} = 3$

(g) $26T_{n+1} = 20T_n + 3$ (k) $8T_{n+1} = 2T_n + 3$

Collected numerical results:

time t	T_{exact}	$T_{(d)}$	$T_{(e)}$	$T_{(f)}$	$T_{(g)}$	$T_{(h)}$	$T_{(i)}$	$T_{(j)}$	$T_{(k)}$
0	0	0	0	0	0				
0.1	0.1296	.150	.130	.125	.115				
0.2	0.2256	.255	.227	.219	.204				
0.3	0.2967	.329	.298	.289	.272				
0.4	0.3494	.380	.351	.342	.325				
0.5	0.3884	.416	.390	.381	.365				
1.0	0.4751					1.5	.600	.500	.375
2.0	0.4988					-1.5	.480	.500	.469
3.0	0.4999					4.5	.504	.500	.492
4.0	0.5000					-7.5	.499	.500	.498
5.0	0.5000					16.5	.500	.500	.500

12.4-4

(a) From a handbook formula,

$T = 4(1 - e^{-t/3})$. Check by substitution:

$$2[4(1 - e^{-t/3})] + 6[4(\frac{1}{3}e^{-t/3})] \neq 8$$

$$8 - 8e^{-t/3} + 8e^{-t/3} \neq 8 \quad \text{Yes; OK}$$

(b) $\Delta t = 1$; Eq. 12.4-8 becomes ($\beta = \frac{1}{2}$)

$$[\frac{1}{2}2 + 6]T_{n+1} = 8 - [\frac{1}{2}2 - 6]T_n$$

$$7T_{n+1} = 8 + 5T_n \quad \text{or} \quad T_{n+1} = \frac{1}{7}(8 + 5T_n)$$

t	n	T_n	T_{exact}
0	0	0	0
1	1	1.143	1.134
2	2	1.959	1.946
3	3	2.542	2.528
4	4	2.959	2.946
5	5	3.256	3.245
6	6	3.469	3.459
7	7	3.621	3.612
8	8	3.729	3.722

(c) $\Delta t = 10$; Eq. 12.4-8 becomes ($\beta = \frac{1}{2}$)

$$[\frac{1}{2}2 + \frac{1}{10}6]T_{n+1} = 8 - [\frac{1}{2}2 - \frac{1}{10}6]T_n$$

$$1.6T_{n+1} = 8 - 0.4T_n \quad \text{or} \quad T_{n+1} = 5 - \frac{T_n}{4}$$

t	n	T_n	T_{exact}
0	0	0	0
10	1	5.00	3.875
20	2	3.75	3.995
		4.06	4.000
40	4	3.98	4.000
50	5	4.004	4.000
60	6	3.999	4.000
70	7	4.000	4.000
80	8	4.000	4.000

(a) Consider a cube one unit on a side, so $V = 1 \cdot 1 \cdot 1 = 1$

After strains appear in all three coordinate directions, volume is

$$V + dV = (1 + \epsilon_x)(1 + \epsilon_y)(1 + \epsilon_z) = 1 + \epsilon_x + \epsilon_y + \epsilon_z + \underbrace{\text{higher order terms}}_{\text{omit}}$$

$$dV = (V + dV) - V = \epsilon_x + \epsilon_y + \epsilon_z = dV/V$$

(b) With Π from Eq. 12.7-6,

$$\delta \Pi = \int_V (p_{,x} \delta p_{,x} + p_{,y} \delta p_{,y} + p_{,z} \delta p_{,z} + \frac{\rho}{g} \ddot{p} \delta p) dV \\ + \int_{S_s} \rho \ddot{u}_n \delta p dS + \int_{S_f} \frac{1}{g} \ddot{p} \delta p dS$$

← (The last integral is a term for surface waves)

Integrate by parts: e.g. for the $p_{,x} \delta p_{,x}$ term,

$$\int_V p_{,x} \delta p_{,x} dV = - \int_V p_{,xx} \delta p dV + \int_S p_{,x} \delta p dS$$

After all three integ. by parts, the integrand of the surface integral thus generated is $p_{,x} l + p_{,y} m + p_{,z} n$ which is $p_{,n}$. Also, $S = S_s + S_f$. Therefore

$$\delta \Pi = - \int_V (p_{,xx} + p_{,yy} + p_{,zz} - \frac{\rho}{g} \ddot{p}) \delta p dV \\ + \int_{S_s} (p_{,n} + \rho \ddot{u}_n) \delta p dS + \int_{S_f} (p_{,n} + \frac{1}{g} \ddot{p}) \delta p dS$$

For $\delta \Pi = 0$, integrands of the three integrals must vanish separately. Thus

$$p_{,xx} + p_{,yy} + p_{,zz} - \frac{\rho}{g} \ddot{p} = 0 \quad \text{in } V$$

as in Eq. 12.7-4

$$p_{,n} + \rho \ddot{u}_n = 0 \quad \text{on } S_s$$

as in Eq. 12.7-5

$$p_{,n} + \frac{1}{g} \ddot{p} = 0 \quad \text{on } S_f$$

as noted above Eq. 12.8-7

$$(c) \quad \Pi = \int \left[\bar{p}_{,x}^2 + \bar{p}_{,y}^2 + \bar{p}_{,z}^2 - \left(\frac{\omega \bar{p}}{c} \right)^2 \right] dV$$

$$\delta \Pi = 2 \int \left[\bar{p}_{,x} \delta \bar{p}_{,x} + \bar{p}_{,y} \delta \bar{p}_{,y} + \bar{p}_{,z} \delta \bar{p}_{,z} - \frac{\omega^2}{c^2} \bar{p} \delta \bar{p} \right] dV$$

Integrate by parts: e.g. the 1st of 3 is

$$\int \bar{p}_{,x} \delta \bar{p}_{,x} dV = - \int \bar{p}_{,xx} \delta \bar{p} dV + \int \bar{p}_{,x} \delta \bar{p} l dS$$

$$\text{Thus } \delta \Pi = 2 \int \left[-\nabla^2 \bar{p} - \frac{\omega^2}{c^2} \bar{p} \right] \delta \bar{p} dV + \int \bar{p}_{,n} \delta \bar{p} dS$$

From Eq. 12.7-5, surface integral vanishes

if $\ddot{u}_n = 0$. Hence, $\delta \Pi = 0$ implies

$$\nabla^2 \bar{p} + \frac{\omega^2}{c^2} \bar{p} = 0$$

12.7-2

(a) Matrices are like $[k]$ and $[m]$ of a truss el. $\left(\frac{A}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \frac{\omega^2 A L}{6c^2} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right) \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$$\text{Let } \lambda = \frac{\omega^2 L^2}{6c^2}; \quad \begin{vmatrix} 1-2\lambda & -1-\lambda \\ -1-\lambda & 1-2\lambda \end{vmatrix} = 3\lambda^2 - 6\lambda = 0$$

$$\lambda = 0 \text{ or } \lambda = 2; \text{ for latter, } \omega^2 = \frac{12c^2}{L^2}, \omega = 3.46 \frac{c}{L}$$

(b) Matrices are like stress stiffness matrix

$[k_s]$ & mass matrix $[m]$ of standard beam el.

$$\text{Set } p_{1,x} = 0 \quad \left(\frac{A}{30L} \begin{bmatrix} 36 & -36 \\ -36 & 36 \end{bmatrix} - \frac{\omega^2 A L}{c^2} \frac{1}{420} \begin{bmatrix} 156 & 54 \\ 54 & 156 \end{bmatrix} \right) \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Let } \lambda = \frac{\omega^2 L^2}{14c^2}; \quad \begin{vmatrix} 6-26\lambda & -6-9\lambda \\ -6-9\lambda & 6-26\lambda \end{vmatrix} = 0$$

$$595\lambda^2 - 420\lambda = 0. \quad \lambda = 0 \text{ or } \lambda = 0.7059.$$

$$\text{For the latter, } \omega^2 = 9.8824 \frac{c^2}{L^2}, \omega = 3.1436 \frac{c}{L}$$

12.7-3

Fig. 12.7-2, with $p_1 = 0$ and $p_5 = 0$. Thus, after combining two of the elements whose matrices appear in Eqs. 12.7-14,

$$\left(\frac{A}{3L} \begin{bmatrix} 16 & -8 & 0 \\ -8 & 14 & -8 \\ 0 & -8 & 16 \end{bmatrix} - \left(\frac{\omega}{c} \right)^2 \frac{AL}{30} \begin{bmatrix} 16 & 2 & 0 \\ 2 & 8 & 2 \\ 0 & 2 & 16 \end{bmatrix} \right) \begin{Bmatrix} p_2 \\ p_3 \\ p_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

For the first mode, $p_2 = p_4$, so

$$\left(\frac{A}{3L} \begin{bmatrix} 16 & -8 \\ -16 & 14 \end{bmatrix} - \frac{\omega^2}{c^2} \left(\frac{AL}{30} \right) \begin{bmatrix} 16 & 2 \\ 4 & 8 \end{bmatrix} \right) \begin{Bmatrix} p_2 \\ p_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Let } \lambda = \frac{\omega^2 L^2}{10c^2}, \text{ then } \begin{vmatrix} 16(1-\lambda) & -8-2\lambda \\ -16-4\lambda & 14-8\lambda \end{vmatrix} = 0$$

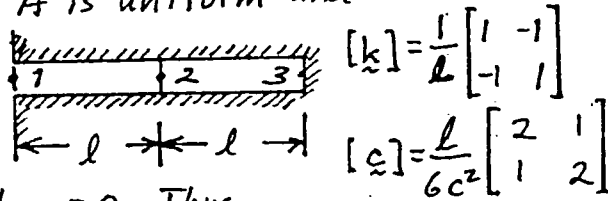
$$\text{from which } 120\lambda^2 - 416\lambda + 96 = 0$$

$$\lambda_1 = \frac{416 - 356.34}{240} = 0.248596$$

$$\omega_1 = 1.5767 \frac{c}{L}$$

12.7-4

(a) A is uniform and cancels out.



Set $p_1 = 0$. Thus

$$\left(\frac{1}{L} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \frac{\omega^2 L}{6c^2} \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \right) \begin{Bmatrix} \bar{p}_2 \\ \bar{p}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let $\lambda = \frac{\omega^2 L^2}{6c^2}$, $\begin{vmatrix} 2-4\lambda & -1-\lambda \\ -1-\lambda & 1-2\lambda \end{vmatrix} = 0$

Yields $7\lambda^2 - 10\lambda + 1 = 0$, $\lambda_1 = 0.10819$

$\omega_1 = \frac{0.8057c}{L} = \frac{1.611c}{L} \quad (l = \frac{L}{2})$

(b) Matrices from Eq. 12.7-14.

A is uniform and cancels out.

$$[k] = \frac{1}{3L} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}, [c] = \frac{L}{30c^2} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

where, in getting $[c]$, we use $\frac{p}{B} = \frac{1}{c^2}$.

For this problem set $p_1 = 0$. Thus

$$\left(\frac{1}{3L} \begin{bmatrix} 16 & -8 \\ -8 & 7 \end{bmatrix} - \frac{\omega^2 L}{30c^2} \begin{bmatrix} 16 & 2 \\ 2 & 4 \end{bmatrix} \right) \begin{Bmatrix} \bar{p}_2 \\ \bar{p}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

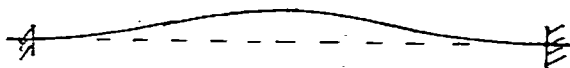
Let $\lambda = \frac{L^2 \omega^2}{10c^2}$, $\begin{vmatrix} 16-16\lambda & -8-2\lambda \\ -8-2\lambda & 7-4\lambda \end{vmatrix} = 0$

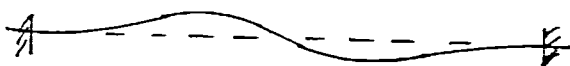
Yields $15\lambda^2 - 52\lambda + 12 = 0$, $\lambda_1 = 0.2486$,

$\omega_1 = \frac{1.5767c}{L}$

$\omega_{\text{exact}} = \frac{\pi c}{2L} = \frac{1.5707c}{L}$

12.8-1

No fluid: 

With fluid: 

12.8-2

In the lower partition of Eq. 12.8-6, $\tilde{M}_F$ and $\tilde{C}_F$ are zero, but $\tilde{W}_F$ appears in place of $\tilde{M}_F$. Thus in place of Eq. 12.8-8 we obtain

$$\tilde{K}_F \tilde{P} + \rho \tilde{S} \ddot{\tilde{D}} + \tilde{W}_F \ddot{\tilde{P}} = 0$$

Substitute $\tilde{D} = \bar{\tilde{D}} \sin \omega t$; thus

$$\tilde{K}_F \tilde{P} - \omega^2 \rho \tilde{S} \bar{\tilde{D}} - \omega^2 \tilde{W}_F \ddot{\tilde{P}} = 0$$

$$\tilde{P} = [\tilde{K}_F - \omega^2 \tilde{W}_F]^{-1} \omega^2 \rho \tilde{S} \bar{\tilde{D}} \quad (A)$$

From the upper partition of Eq. 12.8-6, with $\tilde{D} = \bar{\tilde{D}} \sin \omega t$,

$$\tilde{K} \bar{\tilde{D}} - \tilde{S}^T \tilde{P} - \omega^2 \tilde{M} \bar{\tilde{D}} = 0 \quad (B)$$

Substitute for $\tilde{P}$ from (A); thus (B) yields

$$\left[\tilde{K}_F - \omega^2 \underbrace{\left(\rho \tilde{S}^T [\tilde{K}_F - \omega^2 \tilde{W}_F]^{-1} \tilde{S} + \tilde{M} \right)}_{\text{added mass}} \right] \bar{\tilde{D}} = 0$$

13.1-1

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Yields bilinear element.
Check:

$$u = \left[\frac{1}{4}(1-\xi)(1-\eta) - \frac{N_5}{2} - \frac{N_8}{2} \right] u_1 + N_5 \left(\frac{u_1}{2} + \frac{u_2}{2} \right) +$$

$$\left[\frac{1}{4}(1+\xi)(1-\eta) - \frac{N_5}{2} - \frac{N_6}{2} \right] u_2 + N_6 \left(\frac{u_2}{2} + \frac{u_3}{2} \right) +$$

$$\left[\frac{1}{4}(1+\xi)(1+\eta) - \frac{N_6}{2} - \frac{N_7}{2} \right] u_3 + N_7 \left(\frac{u_3}{2} + \frac{u_4}{2} \right) +$$

$$\left[\frac{1}{4}(1-\xi)(1+\eta) - \frac{N_7}{2} - \frac{N_8}{2} \right] u_4 + N_8 \left(\frac{u_4}{2} + \frac{u_1}{2} \right)$$

$$u = \frac{1}{4} \left[(1-\xi)(1-\eta) u_1 + (1+\xi)(1-\eta) u_2 + (1+\xi)(1+\eta) u_3 + (1-\xi)(1+\eta) u_4 \right] \quad \checkmark$$

13.1-2

If $[K]$ operates on $[u_A \ \theta_A \ u_B \ \theta_B]^T$, $[K] =$

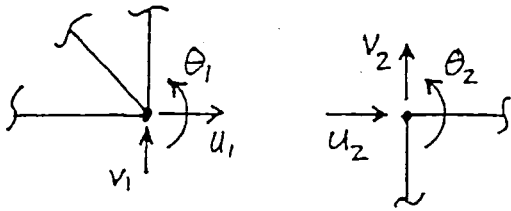
$$\frac{2EI}{L^3} \begin{bmatrix} 6 & 3L & 0 & 0 \\ 3L & 2L^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{2EI}{L^3} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 3L \\ 0 & 0 & 3L & 2L^2 \end{bmatrix} +$$

$$\frac{EI}{L} \begin{bmatrix} A & 0 & -A & 0 \\ 0 & 4I & 0 & 2I \\ -A & 0 & A & 0 \\ 0 & 2I & 0 & 4I \end{bmatrix}, \quad \begin{Bmatrix} u_A \\ \theta_A \\ u_B \\ \theta_B \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[I]} \begin{Bmatrix} u_A \\ \theta_A \\ u_B \\ \theta_B \end{Bmatrix}$$

$[I]^T [K] [I] =$

$$EI \begin{bmatrix} 24/L^3 & 6/L^2 & 6/L^2 \\ 6/L^2 & 8/L & 2/L \\ 6/L^2 & 2/L & 8/L \end{bmatrix}$$

13.1-3



$$\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\substack{[C_r] \\ 2 \times 4}} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\substack{[C_c] \\ 2 \times 2}} \begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \end{Bmatrix} = \{0\}$$

13.1-4

$$\{\underline{D}_r\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \end{Bmatrix}, \quad \{\underline{D}_c\} = \begin{Bmatrix} v_2 \\ u_3 \\ v_3 \end{Bmatrix} \quad \text{Eq. 8.5-6 yields:}$$

$$v_2 = -\frac{a}{b} u_1 + v_1 + \frac{a}{b} u_2$$

$$u_3 = u_1$$

$$v_3 = -\frac{a}{b} u_1 + v_1 + \frac{a}{b} u_2$$

Now write these eqs. in homogeneous form:

$$\begin{bmatrix} a/b & -1 & -a/b & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ a/b & -1 & -a/b & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \{0\}$$

Three eqs. of constraint.

13.1-5

$$(a) \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}, \quad k = \frac{AE}{L}$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} u_1, \quad [1 \ 0] \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = k$$

$$k u_1 = P, \quad u_1 = \frac{P}{k}$$

$$(b) [0 \ 1] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \bar{u}, \quad \underline{T} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \quad \underline{Q} = \bar{u},$$

$$\underline{\xi}_c^{-1} = 1$$

$$\underline{T}^T \underline{K}' \underline{T} = [1 \ 0] \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = k$$

$$\underline{K}' \begin{Bmatrix} \underline{Q} \\ \underline{\xi}_c^{-1} \underline{Q} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} 0 \\ \bar{u} \end{Bmatrix} = \begin{Bmatrix} -k\bar{u} \\ k\bar{u} \end{Bmatrix},$$

$$-\underline{T}^T \underline{K}' \begin{Bmatrix} \underline{Q} \\ \underline{\xi}_c^{-1} \underline{Q} \end{Bmatrix} = k\bar{u}. \quad \text{Final eq. is}$$

$$k u_1 = P + k\bar{u}, \quad \text{from which } u_1 = \frac{P}{k} + \bar{u}$$

13.1-6

$$\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix} \begin{Bmatrix} v_A \\ v_B \\ v_C \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -P \end{Bmatrix} \quad \begin{array}{l} \text{Enforce} \\ v_C = v_A + \frac{v_B - v_A}{L} 2L \\ v_C = 2v_B - v_A \end{array}$$

$$\begin{Bmatrix} v_A \\ v_B \\ v_C \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} v_A \\ v_B \end{Bmatrix} = [\underline{T}] \begin{Bmatrix} v_A \\ v_B \end{Bmatrix} \quad (a)$$

$$[\underline{T}]^T ([\underline{K}][\underline{T}]) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} k \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix} = k \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$$

$$[\underline{T}]^T \{R\} = [\underline{T}]^T \begin{Bmatrix} 0 \\ 0 \\ -P \end{Bmatrix} = \begin{Bmatrix} P \\ -2P \end{Bmatrix}$$

$$k \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \begin{Bmatrix} v_A \\ v_B \end{Bmatrix} = \begin{Bmatrix} P \\ -2P \end{Bmatrix} \quad \text{yields } \begin{Bmatrix} v_A \\ v_B \end{Bmatrix} = \frac{P}{k} \begin{Bmatrix} 1/6 \\ -1/3 \end{Bmatrix}$$

$$\text{Hence Eq. (a) gives } v_C = -\frac{5P}{6k}$$

13.1-7

$u = cx$, where $c = \text{constant}$. But

$u_1 = cx_1$, so $c = \frac{u_1}{x_1}$ and $u = \frac{u_1}{x_1} x$.

$$\text{Then } \begin{aligned} u_2 &= \frac{x_2}{x_1} u_1 \\ u_3 &= \frac{x_3}{x_1} u_1 \end{aligned} \quad \begin{bmatrix} \frac{x_2}{x_1} & -1 & 0 \\ \frac{x_3}{x_1} & 0 & -1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \{0\}$$

13.1-8

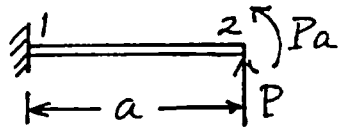
$$\frac{EI}{a^3} \begin{bmatrix} 24 & 0 & -12 & 6a \\ 0 & 8a^2 & -6a & 2a^2 \\ -12 & -6a & 12 & -6a \\ 6a & 2a^2 & -6a & 4a^2 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ P \\ 0 \end{Bmatrix} \quad \text{or } [K] \{D\} = \{R\}$$

$$\begin{Bmatrix} w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & a \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = [T] \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix}$$

$$[T]^T ([K] [T]) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & a & 1 \end{bmatrix} \frac{EI}{a^3} \begin{bmatrix} 12 & -6a \\ -6a & 4a^2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \frac{EI}{a^3} \begin{bmatrix} 12 & -6a \\ -6a & 4a^2 \end{bmatrix}$$

$$[T]^T \{R\} = \begin{Bmatrix} P \\ Pa \end{Bmatrix}$$

$$\frac{EI}{a^3} \begin{bmatrix} 12 & -6a \\ -6a & 4a^2 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} P \\ Pa \end{Bmatrix} \quad \text{yields} \quad \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 5Pa^3/6EI \\ 3Pa^2/2EI \end{Bmatrix}$$

Beam theory: 

$$v_2 = \frac{Pa^3}{3EI} + \frac{(Pa)a^2}{2EI} = \frac{5Pa^3}{6EI}$$

$$\theta_2 = \frac{Pa^2}{2EI} + \frac{(Pa)a}{EI} = \frac{3Pa^2}{2EI}$$

13.1-9

$$\text{For one el., } [k] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Or, if v_1 and v_2 are suppressed,

$$[k] = \frac{2EI}{L} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

For the present two-element structure,

$$\frac{2EI}{L} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ M_0 \end{Bmatrix}$$

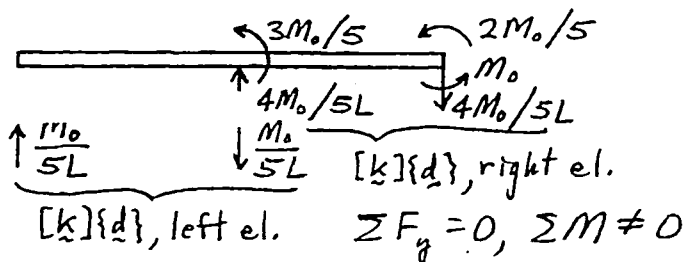
Here $[\theta_1 \ \theta_2 \ \theta_3]^T = [I] \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}$, where

$$[I] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}. \text{ Transformations, as in Eqs. 13.1-4, yield}$$

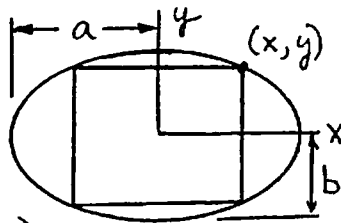
$$\frac{2EI}{L} \begin{bmatrix} 2 & 1 \\ 1 & 8 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ M_0 \end{Bmatrix}, \quad \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \frac{M_0 L}{30EI} \begin{Bmatrix} -1 \\ 2 \end{Bmatrix}$$

$$(b) \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \frac{M_0 L}{30EI} \begin{Bmatrix} -1 \\ 2 \\ 2 \end{Bmatrix} = M_0 \begin{Bmatrix} 0 \\ 3/5 \\ 2/5 \end{Bmatrix}$$

Nodal forces from separate elements can be found from $[k]\{d\}$ of each element, using rows 1 and 3 of $[k]$.
4x4



13.2-1

$$A = 4xy + \lambda \left[\left(\frac{x}{a} \right)^2 + \left(\frac{y}{b} \right)^2 - 1 \right]$$


$$\frac{\partial A}{\partial x} = 0 = 4y + \lambda \frac{2x}{a^2} \quad (a)$$

$$\frac{\partial A}{\partial y} = 0 = 4x + \lambda \frac{2y}{b^2} \quad (b)$$

$$\frac{\partial A}{\partial \lambda} = 0 = \left[\left(\frac{x}{a} \right)^2 + \left(\frac{y}{b} \right)^2 - 1 \right] \quad (c)$$

(a) times x + (b) times y is

$$0 = 4xy + 4xy + 2\lambda \left[\left(\frac{x}{a} \right)^2 + \left(\frac{y}{b} \right)^2 \right]$$

So, in view of (c), $0 = 4xy + \lambda(1)$

$$\lambda = -4xy$$

From (a), $0 = 4y - \frac{8x^2y}{a^2}$, $x = a/\sqrt{2}$

From (b), $0 = 4x - \frac{8xy^2}{b^2}$, $y = b/\sqrt{2}$

$$A = 4xy = 2ab$$

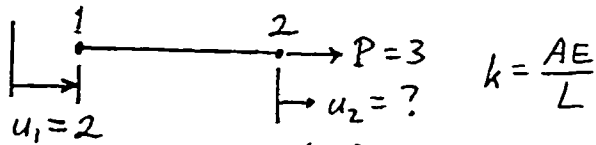
13.2-2

$$[K] = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}, \quad \underbrace{\begin{bmatrix} 1 & -1 \end{bmatrix}}_{[C]} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = 0$$

Eq. 13.2-2 becomes

$$\begin{bmatrix} k & 0 & 1 \\ 0 & k & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ \lambda \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} v_1 \\ v_2 \\ \lambda \end{Bmatrix} = \begin{Bmatrix} P/2k \\ P/2k \\ P/2 \end{Bmatrix}$$

13.2-3



$$[K] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}, \quad \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{[C]} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 2$$

Eq. 13.2-2 becomes

$$\begin{bmatrix} k & -k & 1 \\ -k & k & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \lambda \end{Bmatrix} = \begin{Bmatrix} 0 \\ 3 \\ 2 \end{Bmatrix}, \quad \begin{Bmatrix} u_1 \\ u_2 \\ \lambda \end{Bmatrix} = \begin{Bmatrix} 2 \\ 2 + 3/k \\ 3 \end{Bmatrix}$$

13.2-4

$$(a) \quad \theta_2 = \frac{v_2}{L}, \quad \underbrace{\begin{bmatrix} \frac{1}{L} & -1 \end{bmatrix}}_{[C]} \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = 0$$

$$\begin{bmatrix} 12EI/L^3 & -6EI/L^2 & 1/L \\ -6EI/L^2 & 4EI/L & -1 \\ 1/L & -1 & 0 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \\ \lambda \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \\ 0 \end{Bmatrix}$$

$$v_2 = PL^3/4EI, \quad \theta_2 = PL^2/4EI, \quad \lambda = -P/2$$

$$(b) \quad \theta = [N, x] \{d\} \quad \text{where} \quad \{d\} = [0 \ 0 \ v_2 \ \theta_2]^T$$

$$\theta = \left(\frac{6x}{L^2} - \frac{6x^2}{L^3} \right) v_2 + \left(-\frac{2x}{L} + \frac{3x^2}{L^2} \right) \theta_2. \quad \text{At } x = \frac{L}{2}$$

$$\theta = \theta_c = \frac{1.5}{L} v_2 - \frac{1}{4} \theta_2. \quad \text{Want } \theta_2 = -\frac{1}{2} \theta_c$$

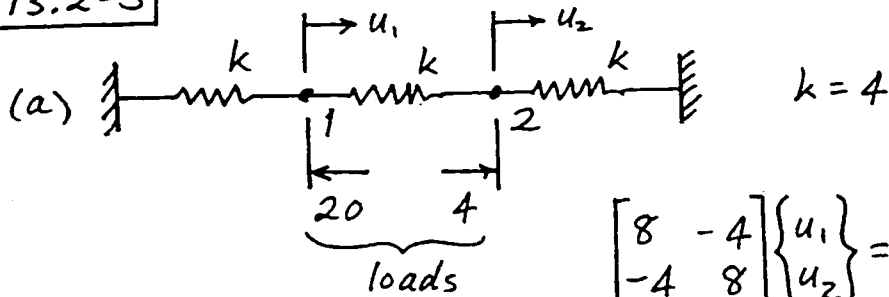
$$\text{Hence } -2\theta_2 = \frac{1.5}{L} v_2 - \frac{1}{4} \theta_2 \quad \text{or } \theta_c = -2\theta_2$$

$$\text{or } \underbrace{\begin{bmatrix} \frac{1.5}{L} & 1.75 \end{bmatrix}}_{[C]} \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = 0$$

$$\begin{bmatrix} 12EI/L^3 & -6EI/L^2 & 1.5/L \\ -6EI/L^2 & 4EI/L & 1.75 \\ 1.5/L & 1.75 & 0 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \\ \lambda \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} v_2 \\ \theta_2 \\ \lambda \end{Bmatrix} = \begin{Bmatrix} 0.03964 PL^3/EI \\ -0.03398 PL^2/EI \\ 0.21363 PL \end{Bmatrix}$$

13.2-5



$$\begin{bmatrix} 8 & -4 \\ -4 & 8 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} -20 \\ 4 \end{Bmatrix}$$

(b) $[\underline{C}] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 0$ where $[\underline{C}] = \begin{bmatrix} 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 8 & -4 & 1 \\ -4 & 8 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \lambda \end{Bmatrix} = \begin{Bmatrix} -20 \\ 4 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} u_1 \\ u_2 \\ \lambda \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.5 \\ -18 \end{Bmatrix}$$

$\lambda = -18$; $|-18|$ is the magnitude of axial constraint force needed at node 1.

(c) $[\underline{C}] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = -1$ where $[\underline{C}] = \begin{bmatrix} 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 8 & -4 & 1 \\ -4 & 8 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \lambda \end{Bmatrix} = \begin{Bmatrix} -20 \\ 4 \\ -1 \end{Bmatrix}, \quad \begin{Bmatrix} u_1 \\ u_2 \\ \lambda \end{Bmatrix} = \begin{Bmatrix} -1 \\ 0 \\ -12 \end{Bmatrix}$$

$\lambda = -12$; $|-12|$ is the magnitude of axial constraint force needed at node 1.

13.3-1

$$(a) \quad \Pi_p = \frac{1}{2} \underline{\underline{D}}^T \underline{\underline{K}} \underline{\underline{D}} - \underline{\underline{D}}^T \underline{\underline{R}} \\ + \frac{1}{2} (\underline{\underline{D}}^T \underline{\underline{C}}^T - \underline{\underline{Q}}^T) \underline{\underline{\alpha}} (\underline{\underline{C}} \underline{\underline{D}} - \underline{\underline{Q}})$$

$$\Pi_p = \frac{1}{2} \underline{\underline{D}}^T \underline{\underline{K}} \underline{\underline{D}} - \underline{\underline{D}}^T \underline{\underline{R}} + \frac{1}{2} (\underline{\underline{D}}^T \underline{\underline{C}}^T \underline{\underline{\alpha}} \underline{\underline{C}} \underline{\underline{D}} - \underline{\underline{D}}^T \underline{\underline{C}}^T \underline{\underline{\alpha}} \underline{\underline{Q}} \\ - \underline{\underline{Q}}^T \underline{\underline{\alpha}} \underline{\underline{C}} \underline{\underline{D}} + \underline{\underline{Q}}^T \underline{\underline{\alpha}} \underline{\underline{Q}})$$

$$\Pi_p = \frac{1}{2} \underline{\underline{D}}^T \underline{\underline{K}} \underline{\underline{D}} - \underline{\underline{D}}^T \underline{\underline{R}} + \frac{1}{2} \underline{\underline{D}}^T \underline{\underline{C}}^T \underline{\underline{\alpha}} \underline{\underline{C}} \underline{\underline{D}} - \underline{\underline{D}}^T \underline{\underline{C}}^T \underline{\underline{\alpha}} \underline{\underline{Q}} + \frac{\underline{\underline{Q}}^T \underline{\underline{Q}}}{2}$$

$$\left\{ \frac{\partial \Pi_p}{\partial \underline{\underline{D}}} \right\} = \underline{\underline{K}} \underline{\underline{D}} - \underline{\underline{R}} + \underline{\underline{C}}^T \underline{\underline{\alpha}} \underline{\underline{C}} \underline{\underline{D}} - \underline{\underline{C}}^T \underline{\underline{\alpha}} \underline{\underline{Q}} = 0$$

$$(\underline{\underline{K}} + \underline{\underline{C}}^T \underline{\underline{\alpha}} \underline{\underline{C}}) \underline{\underline{D}} = \underline{\underline{R}} + \underline{\underline{C}}^T \underline{\underline{\alpha}} \underline{\underline{Q}}$$

(b) Let $[\underline{\underline{\alpha}}] = \alpha [\underline{\underline{\alpha}}']$, where α is a magnitude and $[\underline{\underline{\alpha}}']$ gives proportions of terms in $[\underline{\underline{\alpha}}]$.

$$\text{Then } (\underline{\underline{K}} + \alpha \underline{\underline{C}}^T \underline{\underline{\alpha}}' \underline{\underline{C}}) \underline{\underline{D}} = \underline{\underline{R}} + \alpha \underline{\underline{C}}^T \underline{\underline{\alpha}}' \underline{\underline{Q}}$$

$$\text{or } \left(\frac{1}{\alpha} \underline{\underline{K}} + \underline{\underline{C}}^T \underline{\underline{\alpha}}' \underline{\underline{C}} \right) \underline{\underline{D}} = \frac{1}{\alpha} \underline{\underline{R}} + \underline{\underline{C}}^T \underline{\underline{\alpha}}' \underline{\underline{Q}}$$

$$\text{As } \alpha \rightarrow \infty, \text{ get } (\underline{\underline{C}}^T \underline{\underline{\alpha}}' \underline{\underline{C}}) \underline{\underline{D}} \approx \underline{\underline{C}}^T \underline{\underline{\alpha}}' \underline{\underline{Q}}$$

i.e. $\underline{\underline{Q}}$ dictates response; response associated with $\underline{\underline{K}}$ and $\underline{\underline{R}}$ is lost.

If $\underline{\underline{Q}} = \underline{\underline{0}}$ then $\underline{\underline{D}} = \underline{\underline{0}}$; i.e. the mesh is locked (same conclusion as before).

13.3-2

Constraint $v_1 = v_2$ is $[\underline{C}] \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = 0$,

where $[\underline{C}] = \begin{bmatrix} 1 & -1 \end{bmatrix}$. Eq. 13.3-3 becomes

$$\left(\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} + \alpha \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right) \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}, \text{ Solving,}$$

$$\begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \frac{1}{(k+\alpha)^2 - \alpha^2} \begin{bmatrix} k+\alpha & \alpha \\ \alpha & k+\alpha \end{bmatrix} \begin{Bmatrix} P \\ 0 \end{Bmatrix} = \frac{P}{(k^2 + 2k\alpha)} \begin{Bmatrix} k+\alpha \\ \alpha \end{Bmatrix}$$

$$\text{For } \alpha = 0, \quad \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \frac{P}{k^2} \begin{Bmatrix} k \\ 0 \end{Bmatrix} = \begin{Bmatrix} P/k \\ 0 \end{Bmatrix} \quad \checkmark$$

For $\alpha \rightarrow \infty$,

$$\begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \frac{P}{k \left(\frac{k}{\alpha} + 2 \right)} \begin{Bmatrix} \frac{k}{\alpha} + 1 \\ 1 \end{Bmatrix} \rightarrow \frac{P}{2k} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \checkmark$$

13.3-3

$[C] = [1 \ 0]$. Eq. 13.3-3 becomes

$$\left(\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} + \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \alpha [1 \ 0] \right) \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 3 \end{Bmatrix} + \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \alpha (2)$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{k\alpha} \begin{bmatrix} k & k \\ k & k+\alpha \end{bmatrix} \begin{Bmatrix} 2\alpha \\ 3 \end{Bmatrix} = \begin{bmatrix} \frac{1}{\alpha} & \frac{1}{\alpha} \\ \frac{1}{\alpha} & (\frac{1}{\alpha} + \frac{1}{k}) \end{bmatrix} \begin{Bmatrix} 2\alpha \\ 3 \end{Bmatrix}$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 2 + \frac{3}{\alpha} \\ 2 + \frac{3}{\alpha} + \frac{3}{k} \end{Bmatrix}. \quad \text{For } k=1,$$

	$\alpha=0$	$\alpha=1$	$\alpha=4$	$\alpha=10$	$\alpha=100$	$\alpha=\infty$
u_1	∞	5	2.75	2.30	2.03	2
u_2	∞	8	5.75	5.30	5.03	5

13.3-4

(a) $[C] = \begin{bmatrix} \frac{1}{L} & -1 \end{bmatrix}$. Eq. 13.3-3 becomes

$$\begin{bmatrix} \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} + \alpha \begin{bmatrix} \frac{1}{L^2} & -\frac{1}{L} \\ -\frac{1}{L} & 1 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$

Let $\alpha = \alpha' EI/L$. Thus

$$\begin{bmatrix} 12+\alpha' & -6-\alpha' \\ -6-\alpha' & 4+\alpha' \end{bmatrix} \begin{Bmatrix} v_2 \\ L\theta_2 \end{Bmatrix} = \begin{Bmatrix} PL^3/EI \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \frac{1}{12+4\alpha'} \begin{bmatrix} 4+\alpha' & 6+\alpha' \\ 6+\alpha' & 12+\alpha' \end{bmatrix} \begin{Bmatrix} PL^3/EI \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \frac{PL^3/EI}{\frac{12}{\alpha'} + 4} \begin{Bmatrix} \frac{4}{\alpha'} + 1 \\ - \end{Bmatrix} \xrightarrow{\text{if } \alpha' = \infty} \begin{Bmatrix} PL^3/4EI \\ - \end{Bmatrix}$$

(b) $[C] = \begin{bmatrix} \frac{1.5}{L} & 1.75 \end{bmatrix}$. Eq. 13.3-3 becomes

$$\begin{bmatrix} \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} + \alpha \begin{bmatrix} \frac{2.25}{L^2} & \frac{2.625}{L} \\ \frac{2.625}{L} & 3.0625 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$

Let $\alpha = \alpha' EI/L$. Thus

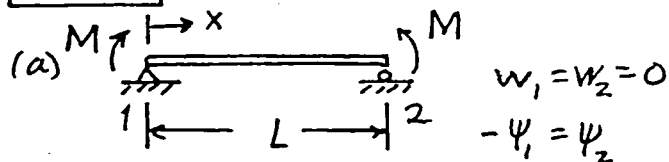
$$\begin{bmatrix} 12+2.25\alpha' & -6+2.625\alpha' \\ -6+2.625\alpha' & 4+3.0625\alpha' \end{bmatrix} \begin{Bmatrix} v_2 \\ L\theta_2 \end{Bmatrix} = \begin{Bmatrix} PL^3/EI \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \frac{1}{12+77.25\alpha'} \begin{bmatrix} 4+3.0625\alpha' & 6-2.625\alpha' \\ 6-2.625\alpha' & 12+2.25\alpha' \end{bmatrix} \begin{Bmatrix} \frac{PL^3}{EI} \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \frac{PL^3/EI}{\frac{12}{\alpha'} + 77.25} \begin{Bmatrix} \frac{4}{\alpha'} + 3.0625 \\ - \end{Bmatrix}$$

$$\text{As } \alpha' \rightarrow \infty, v_2 \rightarrow \frac{3.0625}{77.25} \frac{PL^3}{EI} = 0.03964 \frac{PL^3}{EI}$$

13.4-1



For Mindlin element, $w = 0$ throughout,

and $\psi = \frac{L-x}{L} \psi_1 + \frac{x}{L} \psi_2 = \frac{L-2x}{L} \psi_1$

From Eqs. 13.4-2,

$$U = U_b + U_s = \frac{1}{2} \frac{Ebt^3}{12} \int_0^L \left(\frac{-2\psi_1}{L} \right)^2 dx + \frac{1}{2} \frac{Gbt}{1.2} \int_0^L \left(-\frac{L-2x}{L} \psi_1 \right)^2 dx$$

$$U = \frac{1}{2} \left[\frac{Ebt^3}{12L} 4\psi_1^2 + \frac{Gbt}{1.2} \frac{L\psi_1^2}{3} \right] = \frac{2\psi_1^2}{L} \left[\frac{Ebt^3}{12} + \frac{GbtL^2}{12(1.2)} \right]$$

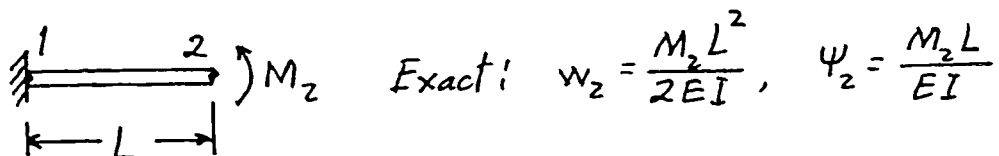
$$\text{Let } I = \frac{bt^3}{12}; \quad U = \frac{2\psi_1^2}{L} EI \left[1 + \frac{GL^2}{1.2Et^2} \right] = \frac{2\psi_1^2}{L} EI_e$$

(b) One-point quadrature: evaluate integrals at $x = \frac{L}{2}$

The second integral (for U_s) vanishes, and

$$U = \frac{2\psi_1^2}{L} \frac{Ebt^3}{12}; \text{ exact; } I = \frac{bt^3}{12}; \quad U = \frac{2\psi_1^2}{L} EI$$

13.4-2



Mindlin beam element, with exact integration:

$$\left(EI \begin{bmatrix} 0 & 0 \\ 0 & 1/L \end{bmatrix} + GA_s \begin{bmatrix} 1/L & -1/2 \\ -1/2 & L/3 \end{bmatrix} \right) \begin{Bmatrix} w_2 \\ \psi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ M_2 \end{Bmatrix}$$

1st eq. gives $\psi_2 = \frac{2}{L} w_2$; then 2nd eq. becomes

$$\frac{EI}{L} \frac{2w_2}{L} + GA_s \left(-\frac{w_2}{2} + \frac{L}{3} \frac{2w_2}{L} \right) = M_2$$

$$w_2 = \frac{M_2}{\left(\frac{2EI}{L^2} + \frac{GA_s}{6} \right)}$$

Set this w_2 equal to $0.9 \frac{M_2 L^2}{2EI}$, thus

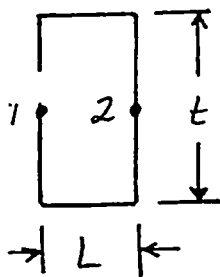
$$0.9 \left(1 + \frac{GA_s L^2}{12EI} \right) = 1$$

$$\frac{GA_s L^2}{12EI} = 0.1111$$

$$\frac{(E/2)(56t/6)L^2}{Ebt^3} = 0.1111$$

$$\left(\frac{L}{t} \right)^2 = 0.2667$$

$$\frac{L}{t} = 0.516, \text{ i.e., about to scale,}$$



13.4-3

$$\epsilon_x = \frac{\partial u}{\partial x} \quad \epsilon_y = \frac{\partial v}{\partial y} \quad u = [N] \{u\} \quad v = [N] \{v\}$$

where the N_i for a rectangular element are given in Eqs. 3.6-4. Thus

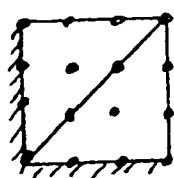
$$\epsilon_x + \epsilon_y = \frac{1}{4ab} \begin{bmatrix} -(b-y), & -(a-x), & (b-y), & -(a+x), & (b+y), & (a+x), & -(b+y), & (a-x) \end{bmatrix} \{d\}$$

Odd powers of x and y integrate to zero, so

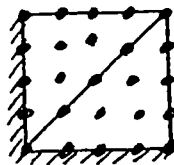
$$\int_{-b}^b \int_{-a}^a (\epsilon_x + \epsilon_y) dx dy = \begin{bmatrix} -b & -a & b & -a & b & a & -b & a \end{bmatrix} \{d\}$$

With one Gauss point at $x=y=0$, the product of Gauss weights is 4, and the Jacobian determinant is $J=ab$. Therefore $(\epsilon_x + \epsilon_y)_0 (4)(ab)$ is the same result.

13.5-1



$$r = \frac{2(9)}{2(6)} \\ = \frac{3}{2}$$



$$r = \frac{2(16)}{2(10)} \\ = \frac{8}{5}$$

13.6-1

(a) From Eq. 13.4-5,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{Bmatrix} = \left(\frac{G}{3} \begin{bmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{bmatrix} + B \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \right) \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \end{Bmatrix}$$

From which $\sigma_x + \sigma_y + \sigma_z = 3B\epsilon_v$

where $\epsilon_v = \epsilon_x + \epsilon_y + \epsilon_z$. Hence

$$B\epsilon_v = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) = \lambda$$

(b) From Eq. 13.4-10,

$$\beta \propto H = \frac{1}{2} \frac{E}{\underbrace{3(1-2\nu)}_B} \{\epsilon\}^T [\underline{E}_B] \{\epsilon\}$$

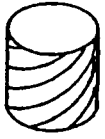
where $[\underline{E}_B]$ is the second square matrix in Eq. 13.4-5. Since $[\underline{E}_B]\{\epsilon\}$ yields volumetric strains ϵ_v ,

$$\beta \propto H = \frac{B}{2} \begin{bmatrix} \epsilon_x & \epsilon_y & \epsilon_z & \gamma_{xy} & \gamma_{yz} & \gamma_{zx} \end{bmatrix} \begin{Bmatrix} \epsilon_v \\ \epsilon_v \\ \epsilon_v \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

where $\epsilon_v = \epsilon_x + \epsilon_y + \epsilon_z$

$$\beta \propto H = \frac{B}{2} \epsilon_v (\epsilon_x + \epsilon_y + \epsilon_z) = \frac{B}{2} \epsilon_v^2$$

14.1-1



Imagine stiff fibers in rubber. Uniform σ_z on ends makes it unwind; i.e. $v \neq 0$. Can make $v = 0$ by adding torque, but then (by statics) $\tau_{\theta z} = 0$. Need both $v = 0$ and $\tau_{\theta z} = 0$ for axisymmetric behavior.

14.2-1

	<u>zero ϵ energy</u>	<u>nonzero ϵ energy</u>
(a) Plane , 1 Gauss pt.	1, 2, 3, 7, 8	4, 5, 6
(b) Plane , 4 Gauss pts.	1, 2, 3	4, 5, 6, 7, 8
(c) Axisym., 1 Gauss pt.	1, 3, 7, 8	2, 3, 4, 5
(d) Axisym., 4 Gauss pts.	1	2 through 8

14.2-2

For a one-radian segment, with $r = r_7 + J\xi$ and $J = \frac{r_3 - r_4}{2}$,

$$\{\underline{r}_e\} = \int [\underline{N}]^T \{\underline{\Phi}\} dS = \int_{-1}^1 \begin{Bmatrix} (\xi^2 - \xi)/2 \\ 1 - \xi^2 \\ (\xi^2 + \xi)/2 \end{Bmatrix} p r J d\xi$$

(a) p is constant:

$$\{\underline{r}_e\} = \frac{p J r_7}{3} \begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} + \frac{p J^2}{3} \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix}$$

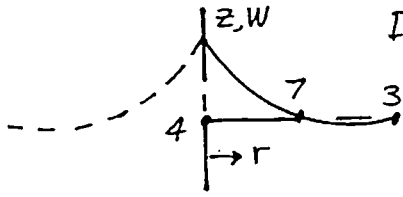
(b)

$$p = \frac{\xi^2 - \xi}{2} p_4 + (1 - \xi^2) p_7 + \frac{\xi^2 + \xi}{2} p_3$$

Algebra is straightforward but tedious

$$\{\underline{r}_e\} = \frac{r_7 J}{30} \begin{Bmatrix} 8p_4 + 4p_7 - 2p_3 \\ 4p_4 + 32p_7 + 4p_3 \\ -2p_4 + 4p_7 + 8p_3 \end{Bmatrix} + \frac{J^2}{30} \begin{Bmatrix} -6p_4 - 4p_7 \\ -4p_4 + 4p_3 \\ 4p_7 + 6p_3 \end{Bmatrix}$$

14.2-3



Implies γ_{rz} dis-
continuous across
 $r=0$; not rea-
sonable; not
compatible.

Need $\frac{dw}{dz} = 0$ at $r=0$, but need such a
nodal d.o.f. to achieve it.

14.2-4

For compatibility, $u=0$ at $r=0$
(no hole appears). Expand u :

$$u = c_1 r + c_2 z + c_3 r^2 + c_4 r z + c_5 z^2 + \dots$$

Hence

$$\epsilon_\theta = \frac{u}{r} = c_1 + c_2 \frac{z}{r} + c_3 r + c_4 z + c_5 \frac{z^2}{r} + \dots$$

$$\epsilon_r = \frac{du}{dr} = c_1 + 2c_3 r + c_4 z + 3c_5 r^2 + \dots$$

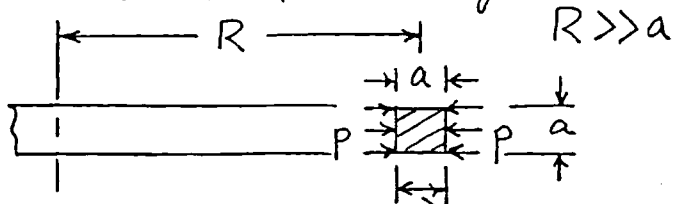
Strains cannot be infinite, so $c_2=0$,
 $c_5=0, \dots$ Hence at $r=0$

$$\left. \begin{aligned} \epsilon_\theta &= c_1 + c_4 z + \dots \\ \epsilon_r &= c_1 + c_4 z + \dots \end{aligned} \right\} \text{same}$$

One could also say that directions
 r and θ lack separate meaning at
 $r=0$, so $\epsilon_\theta = \epsilon_r$ there.

14.2-5

Consider ϵ_r in a ring.



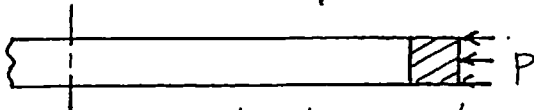
δ = change in this dimension

$$\delta = \frac{PL}{AE} = \frac{(PaRd\theta)a}{(Rd\theta)aE} = \frac{Pa}{E}$$

Per radian, the associated stiffness is

$$k_{r1} = \frac{P}{\delta} = \frac{(Pa)R}{\delta} = ER$$

Consider ϵ_θ in ring of same dimensions.



Δ = radial displacement

$$\sigma_\theta = \frac{PR}{a}, \epsilon_\theta = \frac{\sigma_\theta}{E}, \Delta = R\epsilon_\theta = \frac{PR^2}{Ea}$$

Per radian, the associated stiffness is

$$k_{r2} = \frac{P}{\Delta} = \frac{(Pa)R}{\Delta} = ER \left(\frac{a}{R} \right)^2$$

$$\text{Stiffness ratio: } \frac{k_{r1}}{k_{r2}} = \left(\frac{R}{a} \right)^2$$

Becomes large if R/a is large.

14.2-6

(a) Let $u = a_1 + a_2 r$. Formal process or trial leads to $u = \begin{bmatrix} \frac{r_2 - r}{r_2 - r_1} & \frac{r - r_1}{r_2 - r_1} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

$$\begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \end{Bmatrix} = \begin{Bmatrix} \partial/\partial r \\ 1/r \end{Bmatrix} u = \underbrace{\frac{1}{r_2 - r_1} \begin{bmatrix} -1 & 1 \\ \frac{r_2}{r} - 1 & 1 - \frac{r_1}{r} \end{bmatrix}}_{[B]} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

(b) $[E] = E \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, hence $[B]^T [E] [B]$ is

$$\frac{E}{(r_2 - r_1)^2} \begin{bmatrix} 2 + \frac{r_2^2}{r^2} - 2 \frac{r_2}{r} & -2 + \frac{r_1 + r_2}{r} - \frac{r_1 r_2}{r^2} \\ \text{symm.} & 2 + \frac{r_1^2}{r^2} - 2 \frac{r_1}{r} \end{bmatrix}$$

$$[k] = \int_0^t \int_0^1 \int_{r_1}^{r_2} [B]^T [E] [B] r dr d\theta dz$$

$$[k] = \frac{E t}{(r_2 - r_1)^2} \begin{bmatrix} -(r_2 - r_1)^2 + r_2^2 \ln \frac{r_2}{r_1} & -r_1 r_2 \ln \frac{r_2}{r_1} \\ -r_1 r_2 \ln \frac{r_2}{r_1} & (r_2 - r_1)^2 + r_1^2 \ln \frac{r_2}{r_1} \end{bmatrix}$$

(c) Let $r_m = \frac{1}{2}(r_1 + r_2)$, $L = r_2 - r_1$. With $r = r_m$,

$$[B] = \begin{bmatrix} -1/L & 1/L \\ 1/2r_m & 1/2r_m \end{bmatrix}, [E] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix}$$

$$[k] = [B]^T [E] [B] t r_m (1) L, \text{ i.e.}$$

$$\frac{E t r_m L}{1-\nu^2} \begin{bmatrix} \frac{1}{L^2} - \frac{\nu}{L r_m} + \frac{1}{4 r_m^2} & -\frac{1}{L^2} + \frac{1}{4 r_m^2} \\ -\frac{1}{L^2} + \frac{1}{4 r_m^2} & \frac{1}{L^2} + \frac{\nu}{L r_m} + \frac{1}{4 r_m^2} \end{bmatrix}$$

(d) For $r_m \rightarrow \infty$, part (c) yields

$$[k] = \frac{E t r_m}{(1-\nu^2) L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (A)$$

In $[k]$ of part (b),

$$\ln \frac{r_2}{r_1} = \left(\frac{r_2}{r_1} - 1 \right) - \frac{1}{2} \left(\frac{r_2}{r_1} - 1 \right)^2 + \dots \approx \frac{r_2}{r_1} - 1 = \frac{r_2 - r_1}{r_1} = L/r_1$$

$$[k] = \frac{E t}{L^2} \begin{bmatrix} r_2^2 \frac{L}{r_1} - L^2 & -r_1 r_2 \frac{L}{r_1} \\ -r_1 r_2 \frac{L}{r_1} & r_1^2 \frac{L}{r_1} + L^2 \end{bmatrix} \quad (B)$$

$$\frac{r_2^2}{r_1} L - L^2 = \frac{(r_m + \frac{L}{2})^2}{(r_m - \frac{L}{2})} L - L^2 \approx r_m L - L^2 \approx r_m L$$

$$r_1 r_2 \frac{L}{r_1} = r_2 L \approx r_m L, \quad r_1^2 \frac{L}{r_1} + L^2 = r_1 L + L^2 \approx r_m L$$

Thus (A) & (B) agree if $\nu = 0$.

(e) With $u_1 = 0$, only k_{22} remains, which

$$\text{for } r_1 = 0 \text{ is } k_{22} = \frac{E t}{r_2^2} \left[r_2^2 + \lim_{r_1 \rightarrow 0} \left(r_1^2 \ln \frac{r_2}{r_1} \right) \right]$$

$$\lim_{r_1 \rightarrow 0} \left(r_1^2 \ln \frac{r_2}{r_1} \right) = \lim_{r_1 \rightarrow 0} \frac{\ln r_2 - \ln r_1}{r_1^{-2}} = \lim_{r_1 \rightarrow 0} \frac{-1/r_1}{-2/r_1^3} = 0$$

So k_{22} reduces to $E t$.

$$(f) r_m = \frac{L}{2}, \text{ so } k_{22} = \frac{E t L^2}{2(1-\nu^2)} \left(\frac{1}{L^2} + \frac{2\nu}{L^2} + \frac{1}{L^2} \right)$$

which reduces to $k_{22} = E t$ if $\nu = 0$.

14.2-7

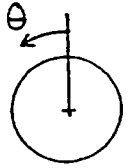
$$\{r_c\} = \int [N]^T \begin{Bmatrix} F_r \\ 0 \end{Bmatrix} dV, \quad F_r = \rho r \omega^2$$

$$\{r_c\} = \frac{\rho \omega^2}{4ab} \int_{-b}^b \int_{-a}^a \int_{-\pi}^{\pi} \begin{Bmatrix} (a-x)(b-y) \\ (a+x)(b-y) \\ (a+x)(b+y) \\ (a-x)(b+y) \end{Bmatrix} r^2 d\theta dx dy$$

where $r = r_m + x$.

14.4-1

(a)



$$\text{old } \theta = \text{new } \theta + \frac{\pi}{2}$$

$$\begin{aligned} \sin n\left(\theta + \frac{\pi}{2}\right) &= \sin n\theta \cos \frac{n\pi}{2} \\ &\quad + \cos n\theta \sin \frac{n\pi}{2} \\ &= \cos n\theta \sin \frac{n\pi}{2} \end{aligned}$$

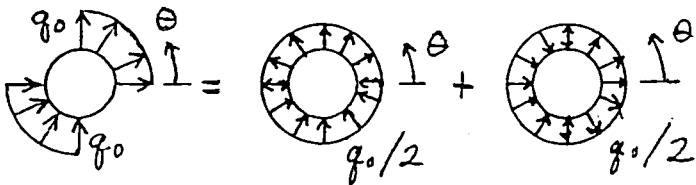
for n odd

Formula in Fig. 6.6-2 becomes

$$q = \frac{4q_0}{\pi} \left(\cos \theta - \frac{\cos 3\theta}{3} + \frac{\cos 5\theta}{5} - \dots \right)$$

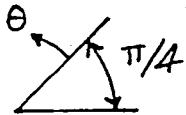
(b)

Fig. 6.6-2 Part (a)



$$\begin{aligned} q &= \frac{2q_0}{\pi} \left(\sin \theta + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \dots \right) \\ &\quad + \frac{2q_0}{\pi} \left(\cos \theta - \frac{\cos 3\theta}{3} + \frac{\cos 5\theta}{5} - \dots \right) \end{aligned}$$

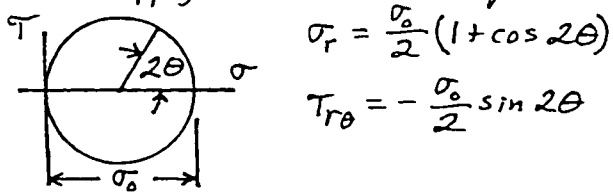
(c) Relocate the $\theta=0$ position to where $\theta = \pi/4$ in part (b), i.e.



14.4-2

Must determine tractions Φ to be applied to the dashed circular boundary.

Can apply Mohr circle analysis.



(a) $\sigma_0 = P/ht$, so

$$\Phi_r = \frac{P}{2ht}(1 + \cos 2\theta), \Phi_\theta = -\frac{P}{2ht} \sin 2\theta, \Phi_z = 0$$

Symmetric terms, zeroth & second harmonics.

Independent of r .

$$(b) \sigma_0 = -\frac{My}{I} = -\frac{M(r \sin \theta)}{\frac{1}{12}th^3} = -\frac{12M}{th^3} r \sin \theta$$

$$(1 + \cos 2\theta) \sin \theta = (2 - 2 \sin^2 \theta) \sin \theta \\ = 2(\sin \theta - \sin^3 \theta)$$

$$\sin 2\theta \sin \theta = 2 \sin \theta \cos \theta \sin \theta \\ = 2 \cos \theta (1 - \cos^2 \theta)$$

$$\Phi_r = -\frac{12M}{th^3} r \sin \theta \frac{1 + \cos 2\theta}{2} = \frac{12M}{th^3} (-r \sin \theta + r \sin^3 \theta)$$

$$\Phi_\theta = -\frac{12M}{th^3} r \sin \theta \left(-\frac{\sin 2\theta}{2}\right) = \frac{12M}{th^3} (r \cos \theta - r \cos^3 \theta)$$

$$\Phi_z = 0$$

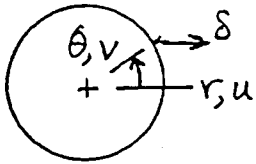
Antisymmetric terms, 1st & 3rd harmonics.

Φ 's on boundary of radius c .

14.4-3

(a) $u = v = 0, w = w_0$

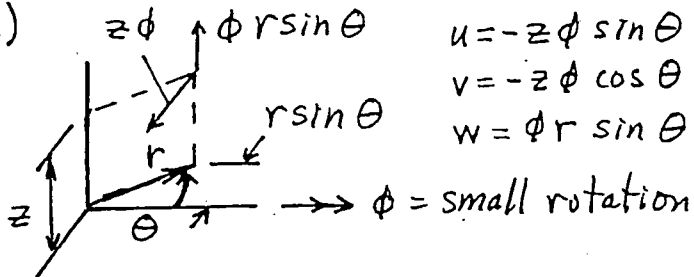
(b)



$$\begin{aligned} u &= \delta \cos \theta \\ v &= -\delta \sin \theta \\ w &= 0 \end{aligned}$$

(c) Orient δ vertically in part (b); then
 $u = \delta \sin \theta, v = \delta \cos \theta, w = 0$

(d)



14.4-4

(a) Shown by direct substitution.

(b) 1st 3 cols., single-barred
2nd 3 cols., double-barred

(c) Consider a_1 thru a_6 in turn. Let x = radial axis along $\theta = 0$, y = radial axis along $\theta = \pi/2$, both in $z=0$ plane.

a_1 — axial (z -dir.) translation.

a_2 — x direction translation

a_3 — rotation about y axis.

a_4 — rotation about z axis.

a_5 — y direction translation.

a_6 — rotation about x axis.

14.4-5

(a) Exact, at center $x = \frac{L}{2}$:

$$v = \frac{5q_0 L^4}{384EI}, \quad M = -\frac{q_0 L^2}{8}$$

$$v_n = \frac{4q_0}{n\pi EI n^4 \pi^4} \frac{L^4}{L} \sin \frac{n\pi x}{L} = \frac{4q_0 L^4}{EI \pi^5 n^5} \sin \frac{n\pi x}{L}$$

$$M_n = EI \frac{d^2 v_n}{dx^2} = -EI \frac{n^2 \pi^2}{L^2} v_n = -\frac{4q_0 L^2}{\pi^3 n^3} \sin \frac{n\pi x}{L}$$

At $x = \frac{L}{2}$ and with n odd,

$$v = \sum v_n = \frac{4q_0 L^4}{EI \pi^5} \left(1 - \frac{1}{3^5} + \frac{1}{5^5} - \dots\right)$$

$$M = \sum M_n = \frac{4q_0 L^2}{\pi^3} \left(-1 + \frac{1}{3^3} - \frac{1}{5^3} + \dots\right)$$

Numerical multipliers in ans. are

	<u>exact</u>	<u>1 term</u>	<u>2 terms</u>	<u>3 terms</u>
v	0.013021	0.013071 (+0.38%)	0.013017 (-0.03%)	0.013021
$ M $	0.125	0.1290 (+3.2%)	0.1242 (-0.6%)	0.1253 (+0.2%)

(b) Exact, at center $x = \frac{L}{2}$:

$$v = \frac{PL^3}{48EI}, \quad M = -\frac{PL}{4}$$

Proceeding as in part (a),

$$v_n = \frac{2PL^3}{EI n^4 \pi^4} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{L}$$

$$\dots \sin \frac{n\pi}{2} \sin \frac{n\pi x}{L}$$

At $x = \frac{L}{2}$, only odd terms are nonzero.

$$v = \sum v_n = \frac{2PL^3}{EI \pi^4} \left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \dots\right)$$

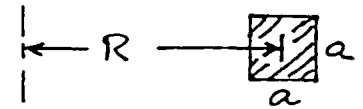
$$M = \sum M_n = \frac{PL}{\pi^2} \left(-1 - \frac{1}{3^2} - \frac{1}{5^2} - \dots\right)$$

Numerical multipliers in ans. are

	<u>exact</u>	<u>1 term</u>	<u>2 terms</u>	<u>3 terms</u>
v	0.02083	0.02053 (-1.45%)	0.02079 (-0.23%)	0.02082 (-0.07%)
$ M $	0.2500	0.2026 (-18.9%)	0.2252 (-9.94%)	0.2333 (-6.69%)

14.5-1

A slender ring of square cross section:



Consider the lowest load harmonic that is statically equivalent to zero:

$$q = q_2^c \cos 2\theta \quad \text{and the associated radial displacement } u = \bar{u}_2 \cos 2\theta$$

Compute $\bar{u}_2$ by equating strain energy U to work W done by load.

$$U = \frac{EI}{2} \int_0^{2\pi} \left[\frac{d^2 u}{(R d\theta)^2} \right]^2 R d\theta = \frac{E}{2} \frac{a^4}{12} \frac{\bar{u}_2^2}{R^3} 4^2 \int_0^{2\pi} \cos^2 2\theta d\theta$$

$$\text{Substitute } \phi = 2\theta: U = \frac{2Ea^4 \bar{u}_2^2}{3R^3} \int_0^{4\pi} \frac{\cos^2 \phi}{2} d\phi = \frac{2\pi Ea^4 \bar{u}_2^2}{3R^3}$$

$$W = \frac{1}{2} \int_0^{2\pi} u (p a R d\theta) = \frac{q_2^c \bar{u}_2 a R}{2} \int_0^{2\pi} \cos^2 2\theta d\theta$$

$$W = \frac{q_2^c \bar{u}_2 a R}{2} \int_0^{4\pi} \frac{\cos^2 \phi}{2} d\phi = \frac{\pi a R q_2^c \bar{u}_2}{2}$$

$$U = W \quad \text{gives} \quad \bar{u}_2 = \frac{3R^4}{4Ea^3} q_2^c$$

Flexural stiffness per radian might be defined as

$$k_{f2} = \frac{R(q_2^c a)}{\bar{u}_2} = \frac{4ER}{3} \left(\frac{a}{R} \right)^4$$

Circumferential stiffness, from Problem 14.2-5, is

$$k_{r2} = ER \left(\frac{R}{a} \right)^2$$

$$\dots \frac{k_{r2}}{k_{f2}} = \frac{3}{4} \left(\frac{R}{a} \right)^2 \quad \text{Very large if } \frac{R}{a} \text{ is large.}$$

14.5-2

(a) First partition, analogous to that in Eq. 14.5-5, is, with $[\underline{\tilde{D}}]$ from Eq. 14.4-4,

$$[\underline{\tilde{D}}] \begin{bmatrix} N_{1,r} \sin n\theta & 0 & 0 \\ 0 & -N_{1,z} \cos n\theta & 0 \\ 0 & 0 & N_{1,z} \sin n\theta \end{bmatrix} =$$

$$\begin{bmatrix} N_{1,r} \sin n\theta & 0 & 0 \\ \frac{N_{1,r}}{r} \sin n\theta & \frac{nN_{1,r}}{r} \sin n\theta & 0 \\ 0 & 0 & N_{1,z} \sin n\theta \\ N_{1,z} \cos n\theta & 0 & N_{1,r} \sin n\theta \\ \frac{nN_{1,z}}{r} \cos n\theta & -(N_{1,r} - \frac{N_{1,z}}{r}) \cos n\theta & 0 \\ 0 & -N_{1,z} \cos n\theta & \frac{nN_{1,z}}{r} \cos n\theta \end{bmatrix}$$

Differs from Eq. 14.5-5 in algebraic signs and sine & cosine terms.

(b) To see form of $[\underline{\tilde{K}}_n]$, part (a) as compared with that provided by Eq. 14.5-5, need consider only sine & cosine terms. Let $c = \cos n\theta$, $s = \sin n\theta$. With $[\underline{\tilde{E}}]$ populated as in Eq. 14.4-3, part (a) yields

$$\begin{bmatrix} s & s & 0 & s & nc & 0 \\ 0 & ns & 0 & 0 & -c & -c \\ 0 & 0 & s & s & 0 & nc \end{bmatrix} \begin{bmatrix} s & ns & s \\ s & ns & s \\ s & ns & s \\ s & ns & s \\ nc & -c & nc \\ nc & -c & nc \end{bmatrix}$$

$$\text{Form of } [\underline{\tilde{B}}]^T [\underline{\tilde{E}}] [\underline{\tilde{B}}]: \begin{bmatrix} s^2 + nc^2 & n(s^2 - c^2) & s^2 + nc^2 \\ n(s^2 - c^2) & ns^2 + c^2 & n(s^2 - c^2) \\ s^2 + nc^2 & n(s^2 - c^2) & s^2 + nc^2 \end{bmatrix}$$

Off-diag. blocks contain $s_i s_j$ & $c_i c_j$ with $i \neq j$; integrate to zero over $\theta = -\pi$ to $\theta = +\pi$. On-diag. blocks integrate to π (or to 2π for $n=0$), for both s^2 and c^2 terms.

In similar fashion, from the first partition in Eq. 14.5-5,

$$\begin{bmatrix} c & c & 0 & c & -ns & 0 \\ 0 & nc & 0 & 0 & s & s \\ 0 & 0 & c & c & 0 & -ns \end{bmatrix} \begin{bmatrix} c & nc & c \\ c & nc & c \\ c & nc & c \\ c & nc & c \\ -ns & s & -ns \\ -ns & s & -ns \end{bmatrix}$$

$$\text{Form of } [\underline{\tilde{B}}]^T [\underline{\tilde{E}}] [\underline{\tilde{B}}] \rightarrow [\underline{\tilde{E}}] [\underline{\tilde{B}}]$$

$$\begin{bmatrix} c^2 + ns^2 & n(c^2 - s^2) & c^2 + ns^2 \\ n(c^2 - s^2) & nc^2 + s^2 & n(c^2 - s^2) \\ c^2 + ns^2 & n(c^2 - s^2) & c^2 + ns^2 \end{bmatrix}$$

Since s^2 and c^2 both integrate to π (with limits $-\pi$ to $+\pi$), the result is the same as produced by part (a).

(c) The middle column of the 6 by 3 result matrix in part (a) changes sign. Then in part (b), row 2 of $[\underline{\tilde{B}}]^T$ and column 2 of $[\underline{\tilde{E}}][\underline{\tilde{B}}]$ change sign. Hence the form of sine and cosine terms in $[\underline{\tilde{B}}]^T [\underline{\tilde{E}}] [\underline{\tilde{B}}]$ becomes

$$\begin{bmatrix} s^2 + nc^2 & -n(s^2 - c^2) & s^2 + nc^2 \\ -n(s^2 - c^2) & ns^2 + c^2 & -n(s^2 - c^2) \\ s^2 + nc^2 & -n(s^2 - c^2) & s^2 + nc^2 \end{bmatrix}$$

Signs of some terms differ, so $[\underline{\tilde{K}}_n]$ changes.

14.5-3

(a) If $n=6$, $\sin n\theta$ goes to 12π when $\theta=2\pi$, i.e. 12 half-waves, so $m=3(12)=36$

(b) For solid-of-revolution analysis, 4-noded

$[k] = [B]^T [E] [B]$. Count multiplications,
 12×12 12×6 6×6 6×12
 ignoring symmetry of $[k]$: 4 Gauss pts./el.,
 20 els., 6 analyses:

$$4 * 20 * 6 * (6 * 12 * 6 + 6 * 12 * 12) = 622,000$$

For 3-D els., 8 node els., $[B]$ is 6×24 .

8 Gauss pts./el., $20 * 36$ els.:

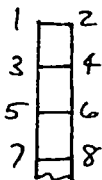
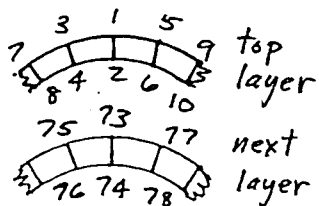
$$8 * 20 * 36 (6 * 24 * 6 + 6 * 24 * 24) = 24.9(10)^6$$

$$\text{ratio} = 24.9(10)^6 / 622,000 = 40$$

(c) 3-D mesh numbering:

Solid of rev.

mesh numbers:



$$3-D: NB^2 = (72 * 21)(78)^2 = 9.2(10)^6$$

$$\text{Solid of rev.} : NB^2 = 42(4)^2 = 672$$

(exclusive of 3 d.o.f. per node in each case).

With 6 solid-of.-rev. analyses,

$$\text{ratio} = 9.2(10)^6 / (6 * 672) = 2300$$

14.5-4

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \frac{1}{r_2 - r_1} \begin{bmatrix} (r_2 - r) \cos n\theta & 0 \\ 0 & (r_2 - r) \sin n\theta \\ (r - r_1) \cos n\theta & 0 \\ 0 & (r - r_1) \sin n\theta \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

From Eq. 10.5-4,

$$\begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \\ \gamma_{r\theta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ \frac{1}{r} & \frac{1}{r} \frac{\partial}{\partial \theta} \\ \frac{1}{r} \frac{\partial}{\partial \theta} & \frac{\partial}{\partial r} - \frac{1}{r} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}. \text{ Hence}$$

$$\begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \\ \gamma_{r\theta} \end{Bmatrix} = \frac{1}{r_2 - r_1} \begin{bmatrix} -\cos n\theta & 0 \\ \frac{r_2 - r}{r} \cos n\theta & \frac{r_2 - r}{r} n \cos n\theta \\ -\frac{r_2 - r}{r} n \sin n\theta & -\frac{r_2}{r} \sin n\theta \\ \cos n\theta & 0 \\ \frac{r - r_1}{r} \cos n\theta & \frac{r - r_1}{r} n \cos n\theta \\ -\frac{r - r_1}{r} n \sin n\theta & \frac{r_1}{r} \sin n\theta \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

15.1-1

(a) Let $\sigma_x = kz$, where $k = \text{constant}$ (σ_x linear in z). Hence

$$M_x = \int \sigma_x z dz = k \int_{-t/2}^{t/2} z^2 dz = \frac{kt^3}{12} = \frac{\sigma_x t^3}{12z},$$

$$\sigma_x = \frac{M_x z}{t^3/12}. \quad \text{At } z = \pm \frac{t}{2}, \sigma_x = \pm \frac{6M_x}{t^2}$$

(b) If τ_{yz} parabolic, $\tau_{yz} = k\left(\frac{t^2}{4} - z^2\right)$
where $k = \text{constant}$.

$$Q_y = \int_{-t/2}^{t/2} \tau_{yz} dz = k \left(\frac{t^2}{4} z - \frac{z^3}{3} \right) \Big|_{-t/2}^{t/2} = k \frac{t^3}{3},$$

$$k = 6Q_y/t^3, \quad \tau_{yz} = \frac{6Q_y}{t^3} \left(\frac{t^2}{4} - z^2 \right).$$

$$\text{At } z=0, \tau_{yz} = \frac{6Q_y}{t^3} \frac{t^2}{4} = 1.5 \frac{Q_y}{t}$$

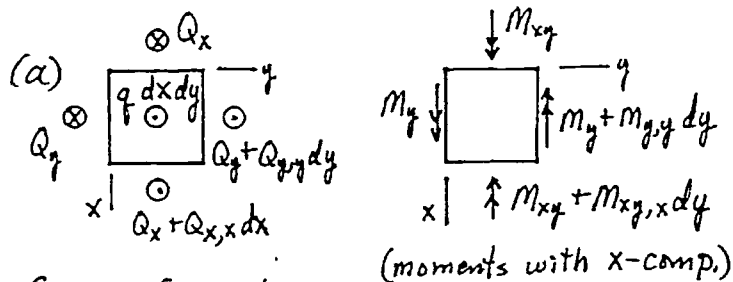
15.1-2

As shown in Prob. 15.1-1,

$$\sigma_x = \frac{M_x z}{t^3/12} \quad \sigma_y = \frac{M_y z}{t^3/12} \quad \tau_{xy} = \frac{M_{xy} z}{t^3/12} \quad \sigma_n = \frac{M_n z}{t^3/12}$$

Substitute these stresses into the σ_n expression given in the problem statement and cancel the common factor $\frac{z}{t^3/12}$. Thus

$$M_n = \frac{1}{2}(M_x + M_y) + \frac{1}{2}(M_x - M_y) \cos 2\theta + M_{xy} \sin 2\theta$$



Sum $\approx$ forces:

$$-Q_y dx - Q_x dy + (Q_x + Q_{x,x}dx)dy + (Q_y + Q_{y,y}dy)dx + q dx dy = 0 ; \text{ yields } Q_{x,x} + Q_{y,y} + q = 0$$

Sum moments about line $y = dy/2$:

$$0 = -M_y dx - M_{xy} dy + (M_y + M_{y,y}dy)dx + (M_{xy} + M_{xy,x}dx)dy - Q_y dx \frac{dy}{2} - (Q_y + Q_{y,y}dy)dx \frac{dy}{2}$$

Neglect higher-order term; get

$$M_{y,y} + M_{xy,x} = Q_y$$

Similarly, moments about line $x = dx/2$ yield

$$M_{x,x} + M_{xy,y} = Q_x$$

$$(b) \left. \begin{aligned} Q_{x,x} &= M_{x,xx} + M_{xy,xy} \\ Q_{y,y} &= M_{y,yy} + M_{xy,xy} \end{aligned} \right\} \text{ into } Q_{x,x} + Q_{y,y} + q = 0$$

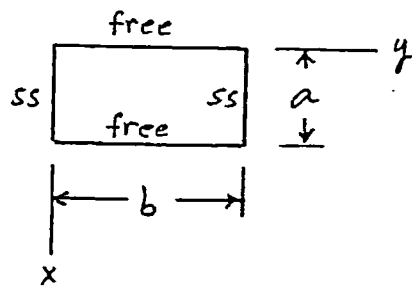
$$\text{Hence } M_{x,xx} + 2M_{xy,xy} + M_{y,yy} + q = 0$$

$$(c) \begin{aligned} M_x &= -D(w_{,xx} + \nu w_{,xy}) \\ M_y &= -D(w_{,yy} + \nu w_{,xx}) \\ M_{xy} &= -D(1-\nu)w_{,xy} \end{aligned}$$

$$-D(w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy}) + q = 0$$

$$\text{or } \nabla^4 w = q/D$$

15.1-4



Bending to a cylindrical surface.

Apply beam theory, with uniformly distributed load $q(l)$ per unit length.

$$w = \frac{5qb^4}{384D}, D = \frac{Et^3}{12(1-\nu^2)}, w = \frac{qb^4(1-\nu^2)}{6.4Et^3}$$

Bending moments at center are

$$M_y = \frac{qb^2}{8}, M_x = \nu \frac{qb^2}{8}, M_{xy} = 0$$

$$\sigma_1 = \sigma_y = \frac{6M_y}{t^2} = \frac{3qb^2}{4t^2}$$

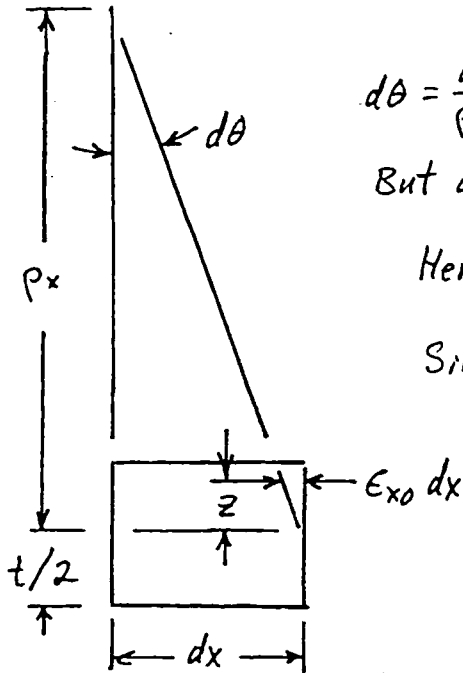
$$\sigma_2 = \sigma_x = \frac{6M_x}{t^2} = \nu \frac{3qb^2}{4t^2}$$

$$\sigma_3 = \sigma_z = 0$$

15.1-5

The given temperature field is $T = -2T_0 \frac{z}{t}$

so initial strains are $\epsilon_{x0} = \epsilon_{y0} = \alpha T = -2\alpha T_0 \frac{z}{t}$



$$d\theta = \frac{dx}{\rho_x}$$

$$\text{But also } d\theta = \frac{\epsilon_{x0} dx}{z} = -\frac{2\alpha T_0}{t} dx$$

$$\text{Hence } \kappa_{ox} = \frac{1}{\rho_x} = -\frac{2\alpha T_0}{t}$$

$$\text{Similarly } \kappa_{oy} = -\frac{2\alpha T_0}{t}$$

$$\{\tilde{\kappa}_0\} = -\frac{2\alpha T_0}{t} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$$

15.1-6

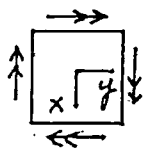
From Eqs. 15.1-4, with $\{K_0\} = \{0\}$,

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = -D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} w_{,xx} \\ w_{,yy} \\ 2w_{,xy} \end{Bmatrix}$$

(a) $w_{,xx} = 2c_1$, $w_{,yy} = 2c_1$, $w_{,xy} = 0$

$M_x = M_y = -2(1+\nu)Dc_1$

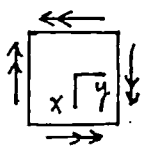
$M_{xy} = 0$



(b) $w_{,xx} = -2c_2$, $w_{,yy} = 2c_2$, $w_{,xy} = 0$

$M_x = 2Dc_2(1-\nu)$

$M_y = -2Dc_2(1-\nu)$ $M_{xy} = 0$



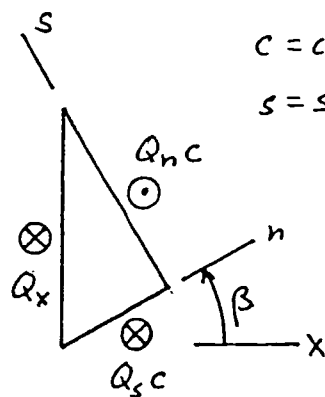
15.1-7

Error proportional to $\frac{1}{1-\nu^2}$, so error usually less than 10% (deflections). In bending to a cylindrical surface with largest stress σ_a , transverse stress is $\sigma_z = 0$ in beam theory but $\sigma_z = \nu \sigma_a$ in plate theory.

Must ask how much error is allowed. If unable to decide, could use 3-D elements (expensive), which should be between beam and plate analyses.

Theoretical studies offer correction factor based on width-to-thickness ratio of beam; see W.C. Young, Roark's Formulas for Stress and Strain.

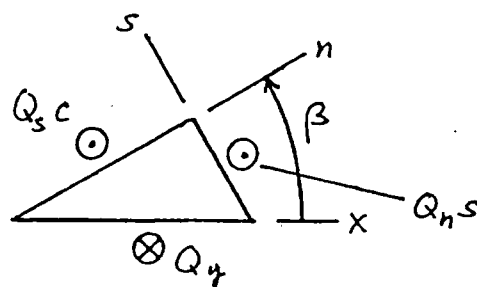
15.1-8



$$c = \cos \beta$$

$$s = \sin \beta$$

$$-Q_x - Q_s c + Q_n c = 0 \quad (a)$$



$$-Q_y + Q_s c + Q_n s = 0 \quad (b)$$

Together, (a) and (b) are

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{Bmatrix} Q_n \\ Q_s \end{Bmatrix}$$

$$\text{or } \{\underline{Q}\} = [\underline{T}]^T \{\underline{Q}'\}$$

$$[\underline{G}] = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} k t \begin{bmatrix} G_n & 0 \\ 0 & G_s \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

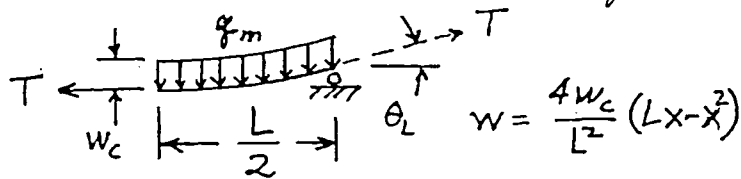
$$[\underline{G}] = k t \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} c G_n & s G_n \\ -s G_s & c G_s \end{bmatrix} = k t \begin{bmatrix} c^2 G_n + s^2 G_s & c s (G_n - G_s) \\ c s (G_n - G_s) & s^2 G_n + c^2 G_s \end{bmatrix}$$

(Assumes that k is direction-independent.)

If $G_n = G_s$, reduces to $[\underline{G}] = k t G \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

15.1-9

(a) Relate tension T to load q_m .



Moments about right end:

$$T w_c - \left(q_m \frac{L}{2} \right) \frac{L}{4} = 0; \quad T = \frac{q_m L^2}{8 w_c} \quad (a)$$

Or, $q_m \frac{L}{2} = T \theta_L$ where θ_L is dw/dx at $x=L$. Thus $q_m \frac{L}{2} = T \frac{4 w_c}{L}$; $T = \frac{q_m L^2}{8 w_c}$

Length change = δ

$$\delta = \frac{1}{2} \int_0^L \left(\frac{dw}{dx} \right)^2 dx = \frac{1}{2} \frac{16 w_c^2}{L^4} \int_0^L (L^2 - 4Lx + 4x^2) dx$$

$$\delta = \frac{8 w_c^2}{L^4} \frac{L^3}{3} = \frac{8 w_c^2}{3 L}, \quad \text{strain} = \epsilon = \frac{\delta}{L}$$

$$T = AE \epsilon = b t E \frac{8 w_c^2}{3 L^2}$$

Substitute from (a) to eliminate T

$$\text{Hence } \frac{q_m L^2}{8 w_c} = \frac{8 E b t w_c^2}{3 L^2}$$

$$q_m = \frac{64 E b t w_c^3}{3 L^4} = \frac{64 E b t^4}{3 L^4} \left(\frac{w_c}{t} \right)^3$$

(b) Center deflection, bending alone, is

$$\frac{5 q_m L^4}{384 E I} \quad \text{so } q_b = \frac{384 E I}{5 L^4} w_c$$

$$q_b + q_s = \frac{384 E (b t^4 / 12)}{5 L^4} \left(\frac{w_c}{t} \right) + \frac{64 E b t^4}{3 L^4} \left(\frac{w_c}{t} \right)^3$$

$$q_b + q_s = \frac{E b t^4}{L^4} \left[6.40 \left(\frac{w_c}{t} \right) + 21.3 \left(\frac{w_c}{t} \right)^3 \right]$$

For $\frac{w_c}{t} = 0.5$,

$$q_b + q_s = \frac{E b t^4}{L^4} [3.20 + 2.67]$$


 roughly equal

15.2-1

(a) Should have w_y linear in x if it is to depend on only w_y nodal values at nodes 3 and 4. But, from Eq. 15.2-4,
 $w_y = [0, 0, 1, 0, x, 2y, 0, x^2, 2xy, 3y^2, x^3, 3xy^2] \{a\}$
 $(w_y)_{y=b} = [0, 0, 1, 0, x, 2b, 0, x^2, 2bx, 3b^2, x^3, 3b^2x] \{a\}$

This edge slope is a cubic in x - needs 4 d.o.f. to define. Hence it must depend on some nodal d.o.f. other than w_y at nodes 3 & 4. It would be only fortuitous (or constant curvature case) if the "other" d.o.f. are such that w_y matches between adjacent els.

(b) On $y = b$, $w = [\text{cubic in } x] \{a\}$
 Requires 4 d.o.f. to define; they are (presumably) w_3, w_4, w_{x3}, w_{x4} . These same d.o.f. used in adjacent el. to define w of same edge. Hence, same w ; compatible. And, if w same, so is $w_{,x}$.

(c) Then w would be quartic on all edges. Five d.o.f. needed to define quartic; these are more than available as nodal d.o.f. at ends of each edge. Expect that w and both slopes will be incompatible.

15.2-2

(a) Let w_n = edge-normal slope.

First arrangement of d.o.f.:

$w, w_x, w_y, w_{xx}, w_{xy}, w_{yy}$ at 1, 2, 3

w_n at 4, 5, 6

Fifth degree terms: $x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5$.

On $x=0$, w is 5th degree in y . Requires 6 d.o.f. to define and 6 are available

(w, w_y , and w_{yy} at nodes 1 & 3). And, on

$x=0$, w_x is 4th degree in y . Requires 5

d.o.f. to define and 5 are available (w_x and w_{xy} at nodes 1 & 3, w_x at node 6).

Compatibility expected.

Second arrangement of d.o.f.:

w, w_x, w_y at all nodes, also w_{xy} at

On $x=0$, 6 d.o.f. needed & available to

define w (w & w_y at 1, 3, 6). Also, 5 d.o.f.

needed & available to define w_x (w_x &

w_{xy} at 1 & 3, w_x at 6). Compat. expected.

(b) D.o.f. w, w_x, w_y at corner nodes, w and w_n at side nodes.

On $x=0$, 6 d.o.f. needed & available to define w (w at nodes 1, 3, 8, 9 and w_y at

nodes 1 & 3). 5 d.o.f. needed to define w_x

but only 4 available on $x=0$ (w_x at 1, 3, 8, 9). Compatibility expected for w

and w_y but not for w_x .

15.2-3

From standard cubic beam functions, written i.t.o. y rather than x ,

$$w_{,y} = [N_{,y}] \{d\}$$

$$w_{,y} = \begin{bmatrix} -\frac{6y}{L_{23}} + \frac{6y^2}{L_{23}^2} & 1 - \frac{4y}{L_{23}} + \frac{3y^2}{L_{23}^2} & \frac{6y}{L_{23}} - \frac{6y^2}{L_{23}^2} & -\frac{2y}{L_{23}} + \frac{3y^2}{L_{23}^2} \end{bmatrix} \begin{Bmatrix} w_2 \\ w_{,y2} \\ w_3 \\ w_{,y3} \end{Bmatrix}$$

$$\text{At } y = \frac{L_{23}}{2}, \quad w_{,y} = w_{,y5} = \begin{bmatrix} -\frac{3}{2L_{23}} & -\frac{1}{4} & \frac{3}{2L_{23}} & -\frac{1}{4} \end{bmatrix} \begin{Bmatrix} w_2 \\ w_{,y2} \\ w_3 \\ w_{,y3} \end{Bmatrix} \quad (A)$$

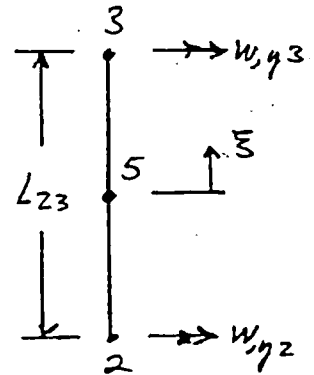
Quadratic variation of ψ_y : can use shape functions from Sec. 6.1.

$$\psi_y = \frac{1}{2}(-\xi + \xi^2)w_{,y2} + (1 - \xi^2)w_{,y5} + \frac{1}{2}(\xi + \xi^2)w_{,y3}$$

Substitute from (A):

$$\psi_y = \frac{3}{2L_{23}}(1 - \xi^2)(w_3 - w_2) + \left(-\frac{1}{2}\xi + \frac{1}{2}\xi^2 - \frac{1}{4} + \frac{1}{4}\xi^2\right)w_{,y2} + \left(\frac{1}{2}\xi + \frac{1}{2}\xi^2 - \frac{1}{4} + \frac{1}{4}\xi^2\right)w_{,y3}$$

$$\psi_y = \frac{3}{2L_{23}}(1 - \xi^2)(w_3 - w_2) + \frac{1}{4}(-1 - 2\xi + 3\xi^2)w_{,y2} + \frac{1}{4}(-1 + 2\xi + 3\xi^2)w_{,y3}$$



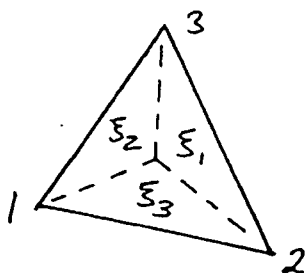
Shear constraint: will enforce $w_{,y2} = \psi_2$ and $w_{,y3} = \psi_3$.

- (a) Not changed: all concentrated center load cases, since the formula would create nodal moment loads only from distributed load. Also, the $N=1$ case of distributed load on a clamped plate, since w at the plate center is the only d.o.f. not restrained.
- (b) Changed very little: other uniform-load clamped-edge cases, since nodal moment loads whose vectors are parallel to mesh boundaries are discarded at boundary nodes when edge-tangent rotations are suppressed. Other nodal moment loads are probably small, as contributions from connected elements may nearly cancel.
- (c) Noticeably changed: uniformly loaded, simply-supported cases. Edge-tangent moment vectors will appear at boundary nodes and associated with d.o.f. that remain active.

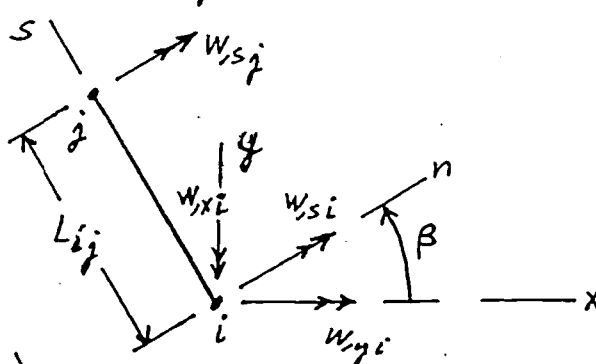
15.2-5

Area coordinates:

ξ_1, ξ_2, ξ_3



Typical side ij :



Along side ij , $w_{ij} = \frac{s(L_{ij}-s)}{2L_{ij}}(w_{si}-w_{sj})$

which gives, at midside, $w_{ijmax} = \frac{L_{ij}}{8}(w_{si}-w_{sj})$

But $w_{si} = w_{yi} \cos \beta - w_{xi} \sin \beta = w_{yi} \frac{y_j - y_i}{L_{ij}} - w_{xi} \frac{x_i - x_j}{L_{ij}}$

Hence $w_{ijmax} = \frac{1}{8}[(y_j - y_i)(w_{yi} - w_{yj}) + (x_i - x_j)(w_{xi} - w_{xj})]$

Use w 's at vertices and midsides to interpolate quadratically over the plate:

$$w = \xi_1 w_1 + \xi_2 w_2 + \xi_3 w_3 + 4(\xi_1 \xi_2 w_{12max} + \xi_2 \xi_3 w_{23max} + \xi_3 \xi_1 w_{31max})$$

$$w = \xi_1 w_1 + \xi_2 w_2 + \xi_3 w_3$$

$$+ \frac{\xi_1 \xi_2}{2} [(y_2 - y_1)(w_{y1} - w_{y2}) + (x_1 - x_2)(w_{x2} - w_{x1})]$$

$$+ \frac{\xi_2 \xi_3}{2} [(y_3 - y_2)(w_{y2} - w_{y3}) + (x_2 - x_3)(w_{x3} - w_{x2})]$$

$$+ \frac{\xi_3 \xi_1}{2} [(y_1 - y_3)(w_{y3} - w_{y1}) + (x_3 - x_1)(w_{x1} - w_{x3})]$$

Can gather terms and revise form if desired.

15.2-6

From Eqs. 15.2-5 and 15.2-12 we see that strains $\{\epsilon\}$ will contain only linear terms. Therefore terms in the stiffness matrix integrand will be no higher than quadratic, and either 3-point formula in Table 7.4-1 will integrate $[k]$ exactly.

15.2-7

(a) Incompatible elements - note that center deflection is overestimated even for concentrated center load, which would not happen with compatible elements.

(b) Uniform load, simply supported, M_c :

Halve h , cut error by factor $\approx \frac{1}{4}$; $2^m = 4$, $m = 2$

Uniform load, clamped, w_c :

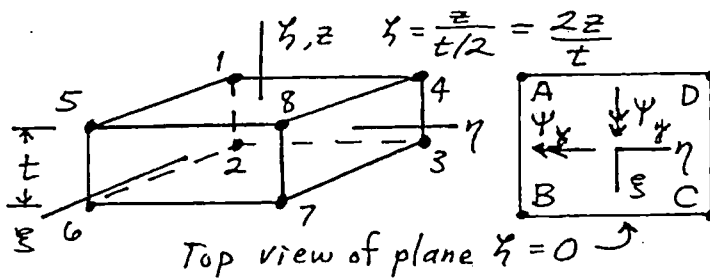
Halve h , cut error by factor $\approx \frac{1}{3}$; $2^m = 3$, $m = 1.6$

(c) w_c : halve h , cut error by factor $\approx \frac{1}{4}$; $2^m = 4$, $m = 2$

M_c : " " " " " " $\approx \frac{1}{4}$; $2^m = 4$, $m = 2$

M_{xy0} : " " " " " " ≈ 0.3 ; $2^m = \frac{1}{0.3}$, $m = 1.7$

15.3-1



(The sketch is simplified; the element need not be rectangular; Ψ_x need not be η -parallel; Ψ_y need not be ξ -parallel.)

$$\begin{aligned} w_1 &= w_2 = w_A & u_2 &= -u_1 = (t/2)\Psi_{xA} \\ w_5 &= w_6 = w_B & u_6 &= -u_5 = (t/2)\Psi_{xB} \\ w_7 &= w_8 = w_C & u_8 &= -u_7 = (t/2)\Psi_{xC} \\ w_3 &= w_4 = w_D & u_4 &= -u_3 = (t/2)\Psi_{xD} \end{aligned}$$

(Formulas for V 's similar.)

Consider e.g. nodes 1 and 2: with $w_1 = w_2$,

$$\frac{1}{8}(1-\xi)(1-\eta)(1+\xi)w_1 + \frac{1}{8}(1-\xi)(1-\eta)(1-\xi)w_2 = \frac{1}{4}(1-\xi)(1-\eta)w_A$$

And with u_1 and u_2 as defined above,

$$\begin{aligned} u &= \frac{1}{8}(1-\xi)(1-\eta)\left(1 + \frac{2z}{t}\right)\left(-\frac{t}{2}\Psi_{xA}\right) \\ &\quad + \frac{1}{8}(1-\xi)(1-\eta)\left(1 - \frac{2z}{t}\right)\left(+\frac{t}{2}\Psi_{xA}\right) + \dots \\ u &= -\frac{z}{4}(1-\xi)(1-\eta)\Psi_{xA} + \dots, \text{ etc.} \end{aligned}$$

15.3-2

$$k = \int \underset{N \times 5}{\underline{B}_m^T} \underset{5 \times 5}{\underline{D}_m} \underset{5 \times N}{\underline{B}_m} dA = \int (\underline{B}_b + \underline{B}_s)^T \underline{D}_m (\underline{B}_b + \underline{B}_s) dA$$

$$k = \int \underline{B}_b^T \underline{D}_m \underline{B}_b dA + \int \underline{B}_s^T \underline{D}_m \underline{B}_s dA \quad (A)$$

$$+ \int \underline{B}_s^T \underline{D}_m \underline{B}_b dA + \int \underline{B}_b^T \underline{D}_m \underline{B}_s dA$$

$$\underline{D}_m = \begin{bmatrix} [3 \times 3] \text{ zero} \\ \text{zero } [2 \times 2] \end{bmatrix} \quad \underline{B} = \underline{B}_b + \underline{B}_s$$

rows 5 & 6 zero first 3 rows zero

$$\underline{D}_m \underline{B}_b: \text{rows 5 \& 6 zero, so } \underline{B}_s^T (\underline{D}_m \underline{B}_b) = 0$$

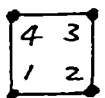
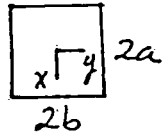
$$\underline{D}_m \underline{B}_s: \text{rows 1, 2, 3 zero, so } \underline{B}_b^T (\underline{D}_m \underline{B}_s) = 0$$

So Eq. (A) reduces to Eq. 15.3-6.

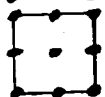
15.3-3

Nodal moments would appear if lateral displacement were created by nodal rotation, but this is not the case for C^0 elements; w depends only on nodal w_i .

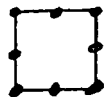
Let $A = 4ab$
 $F_i = \text{nodal force}$



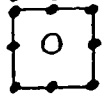
$$F_i = qA/4 \quad (\text{see Fig. 3.11-3})$$



$$\left. \begin{aligned} F_i &= qA/36 \text{ corners} \\ F_i &= qA/9 \text{ midsides} \\ F_i &= 4qA/9 \text{ center} \end{aligned} \right\} (\text{see Problem 6.9-1})$$



$$\left. \begin{aligned} F_i &= -qA/12 \text{ corners} \\ F_i &= qA/3 \text{ midsides} \end{aligned} \right\} (\text{see Fig. 3.11-3})$$



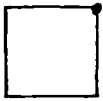
Same as preceding (center node has no w d.o.f.)

15.3-4

Evaluate the number of "free" d.o.f. remaining after boundary conditions have been imposed, then subtract (no. of els.) * (no. of shear constraints per element).

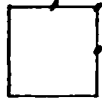
Consider elements in the following order: 4-node, 8-node, 9-node, heterosis.

(a) 1 el./quadrant



$$1 - 2 = -1$$

no d.o.f. left



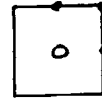
$$5 - 8 = -3$$

no d.o.f. left



$$8 - 8 = 0$$

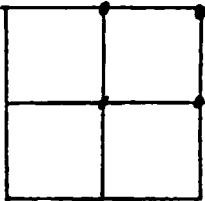
no d.o.f. left



$$7 - 8 = -1$$

no d.o.f. left

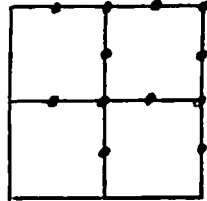
(b)



D.o.f.: $3 + 2(2) + 1 = 8$

$$8 - 4(2) = 0$$

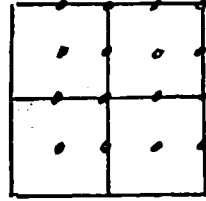
no d.o.f. left



$5(3) + 6(2) + 1 = 28$

$$28 - 4(8) = -4$$

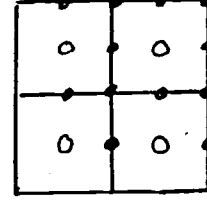
no d.o.f. left



$9(3) + 6(2) + 1 = 40$

$$40 - 4(8) = 8$$

8 d.o.f. left

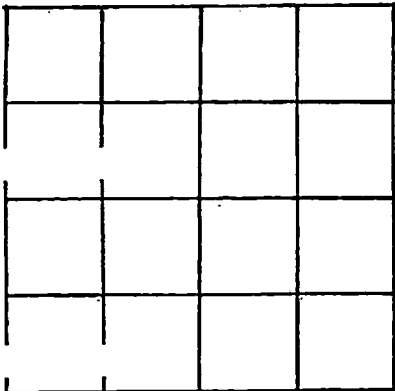


$5(3) + 10(2) + 1 = 36$

$$36 - 4(8) = 4$$

4 d.o.f. left

(c)



4-node el.:

$$9(3) + 6(2) + 1 = 40 \text{ free d.o.f.}$$

$$40 - 16(2) = 8 \text{ d.o.f. left}$$

8-node el.:

$$33(3) + 14(2) + 1 = 128 \text{ free d.o.f.}$$

$$128 - 16(8) = 0 \text{ (no d.o.f. left)}$$

9-node el.:

$$128 + 16(3) = 176 \text{ free d.o.f.}$$

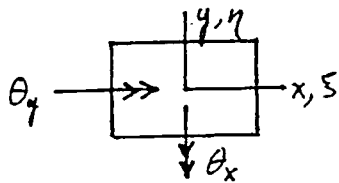
$$176 - 16(8) = 48 \text{ d.o.f. left}$$

Heterosis

$$128 + 16(2) = 160 \text{ free d.o.f.}$$

$$160 - 16(8) = 32 \text{ d.o.f. left}$$

15.3-5



Shear strains
incorrect except
as follows:

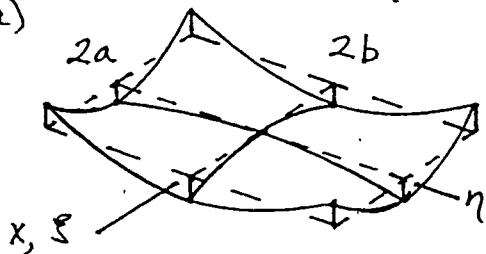
Both γ_{yz} and γ_{zx} correctly evaluated (as zero) when there is rigid body motion ($w = a_0 + a_1x + a_2y$), bending to a cylindrical surface, or constant twist, as then M 's do not vary with x or y . Also correct if nodal rotations are zero and nodal w 's create a state of pure transverse shear strain, uniform over the element.

11.3-6

$$w = 3\xi^2\eta^2 - \xi^2 - \eta^2$$

ξ	η	w
0	0	0
± 1	0	-1
0	± 1	-1
± 1	± 1	1

(a)



(b) With $\Psi_x = \Psi_y = 0$, Transverse shear strains are produced by w alone.

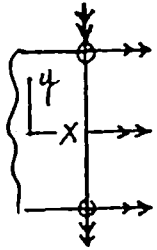
$$\gamma_{zx} = w_{,\xi} = \frac{w_{,\xi}}{a} = \frac{1}{a}(6\xi\eta^2 - 2\xi) = \frac{2\xi}{a}(3\eta^2 - 1)$$

$\gamma_{zx} = 0$ at $\eta = \pm \frac{1}{\sqrt{3}}$. In similar fashion we can show that $\gamma_{yz} = 0$ at $\xi = \pm \frac{1}{\sqrt{3}}$.

(c) Supports only at midsides or only at corners; the latter is more likely for a mesh. Any loads that tend to bend solid lines in sketch into the shape shown.

15.3-9

Two transverse shear strains (e.g. γ_{yz} and γ_{zx}) are set to zero at each of 4 Gauss points. Thus we can eliminate $2 \times 4 = 8$ d.o.f. such as lateral deflection and normal rotation at side nodes. This leaves, e.g. along a y -parallel as shown



Thus, w and edge-normal rotation are linear in y but edge-tangent rotation is quadratic in y , so a moment M_y linear in y can be represented.

15.4-1

$$w = \frac{L-x}{L} w_1 + \frac{x}{L} w_2$$

$$\psi = \frac{L-x}{L} \psi_1 + \frac{x}{L} \psi_2$$

$$\psi_{,x} = -\frac{1}{L} \psi_1 + \frac{1}{L} \psi_2 = \frac{1}{L} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}$$

$$U_b = \frac{EI}{2L^2} \psi_{,x}^T \psi_{,x} L = \frac{EI}{L} \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}^T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix} = \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}^T \underbrace{\frac{EI}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{[k_b]} \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}$$

(Expand $[k_b]$ to 4×4 by adding zeros corresponding to w_1 and w_2 d.o.f.)

$$\gamma_{zx} = w_{,x} - \psi = \begin{bmatrix} -\frac{1}{L} & -\frac{L-x}{L} & \frac{1}{L} & -\frac{x}{L} \end{bmatrix} \begin{Bmatrix} w_1 \\ \psi_1 \\ w_2 \\ \psi_2 \end{Bmatrix}$$

$\nwarrow \{d\}$

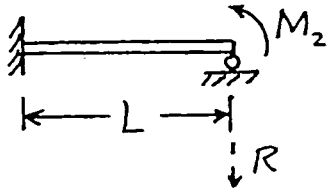
Evaluate γ_{zx} at $x = \frac{L}{2}$ for one-point integration; then

$$U_s = \frac{GA_s}{2} \{d\}^T \begin{Bmatrix} -1/L \\ -1/2 \\ 1/L \\ -1/2 \end{Bmatrix} \begin{bmatrix} -1/L & -1/2 & 1/L & -1/2 \end{bmatrix} \{d\} L$$

$$U_s = \frac{1}{2} \{d\}^T GA_s \begin{bmatrix} 1/L & 1/2 & -1/L & 1/2 \\ 1/2 & L/4 & -1/2 & L/4 \\ -1/L & -1/2 & 1/L & -1/2 \\ 1/2 & L/4 & -1/2 & L/4 \end{bmatrix} \{d\} = \frac{1}{2} \{d\}^T [k_s] \{d\}$$

15.4-2

(a)



Beam theory: zero tip deflection, so

$$\frac{M_2 L^2}{2EI} - \frac{RL^3}{3EI} = 0 ; R = \frac{3M_2}{2L}$$

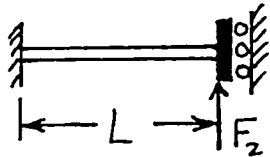
$$\psi_2 = \frac{M_2 L}{EI} - \frac{RL^2}{2EI} = \frac{M_2 L}{4EI}$$

From last d.o.f. in Eq. 15.4-4:

$$\left[\frac{EI}{L} + \frac{L/4}{\left(\frac{L^2}{12EI} + \frac{1}{GA_s} \right)} \right] \psi_2 = M_2$$

Reduces to $\frac{12EI + 4GA_s L^2}{12L + \frac{GA_s L^3}{EI}} \psi_2 = M_2$ $\begin{cases} \text{Small } GA_s, \frac{EI}{L} \psi_2 = M_2 \\ \text{Large } GA_s, \frac{4EI}{L} \psi_2 = M_2 \end{cases}$ ✓

(b)



Beam theory: $w_2 = 2 \frac{F_2 (L/2)^2}{3EI} = \frac{F_2 L^3}{12EI}$

From next-to-last d.o.f. in Eq. 15.4-4:

$$\frac{1}{L} \frac{1}{\left(\frac{L^2}{12EI} + \frac{1}{GA_s} \right)} w_2 = F_2$$

$$\frac{1}{L} \frac{12EIGA_s}{GA_s L^2 + 12EI} w_2 = F_2$$

Small GA_s , $w_2 = \frac{F_2 L}{GA_s}$ ✓

Large GA_s , $w_2 = \frac{F_2 L^3}{12EI}$ ✓

$$15.4-3 \quad x = \frac{L}{2} \xi$$

The only nonzero d.o.f. is w_2 , so $\psi_{,x} = 0$ throughout. With $\frac{d}{dx} = \frac{2}{L} \frac{d}{d\xi}$,

$$\gamma_{zx} = w_{,x} - \psi = w_{,x} = \frac{d}{dx}(1 - 3\xi^2)w_2 = \frac{2}{L}(-2\xi)w_2 = -\frac{4\xi}{L}w_2$$

There is only one stiffness coefficient. Gauss point locations of an order 2 rule are at $\xi = \pm 1/\sqrt{3}$, and weights are each unity.

$$k = GA_s \int_{-1}^1 \left(-\frac{4\xi}{L}\right)^2 \frac{L}{2} d\xi = \frac{8GA_s}{L} \left[\left(-\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 \right] = \frac{16GA_s}{3L}$$

The consistent load at node 2 is $\frac{2qL}{3}$, so

$$\frac{16GA_s}{3L} w_2 = \frac{2qL}{3}, \quad w_2 = \frac{qL^2}{8GA_s}$$

We want to include the exact bending deflection $\frac{qL^4}{384EI}$

$$\frac{qL^2}{8GA^*} = \frac{qL^2}{8GA_s} + \frac{qL^4}{384EI}, \quad \text{hence} \quad \frac{1}{GA^*} = \frac{1}{GA_s} + \frac{L^2}{48EI}$$

15.6-1

Assume that all of the plate is in contact; solve as usual. Where the solution shows an upward deflection (with downward load), eliminate the foundation there in the next solution. Repeat until convergence. It may be necessary to re-introduce foundation in some places where it was previously removed, when a reanalysis shows renewed contact.

16.1-1

(a) Cylindrical part; x-direction axial:

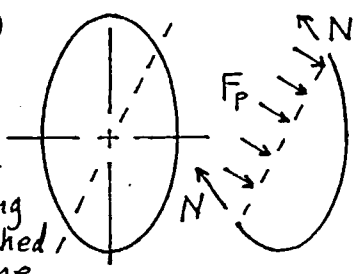
$$\sigma_x = \frac{pR}{2t} \quad \sigma_y = \frac{pR}{t} \quad \tau_{xy} = 0$$

$$N_x = \sigma_x t = \frac{pR}{2}, \quad N_y = \sigma_y t = pR, \quad N_{xy} = 0$$

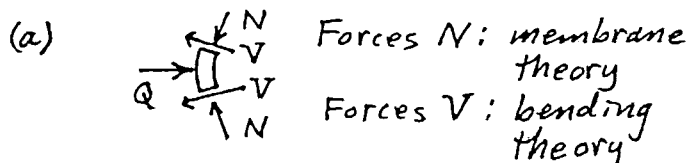
Hemispherical cap: if n and s are any two surface-tangent directions,

$$\sigma_n = \sigma_s = \frac{pR}{2t}, \quad \tau_{ns} = 0$$

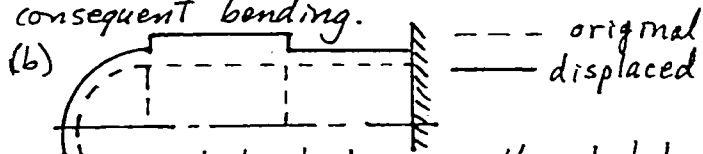
$$N_n = N_s = \frac{pR}{2}, \quad N_{ns} = 0$$

(b)  Membrane forces N not parallel to force F_p from pressure. Transverse shear forces must also appear. Transverse shear forces, and consequent bending, vanish only for circular cylinder.

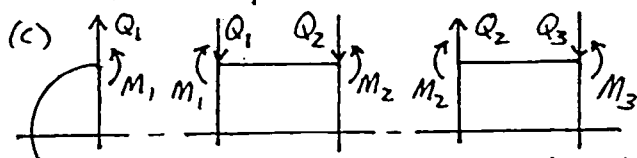
16.1-2



Shrink slice shown to zero size to span only point load Q . Then forces N are parallel and must become infinite to carry Q . Can't happen. Q carried by V , with consequent bending.



Also, but not shown in the sketch, is axial disp. associated with axial ϵ .



Q_i & M_i are line loads uniformly distributed around parallels. Direction of M_2 guessed at. Analysis will show $M_1 = 0$.

16.2-1

$$\epsilon_s = \frac{d}{ds}(\delta_a + \delta_c) + \frac{w}{R} = \frac{d}{ds} \left[u \left(1 + \frac{z}{R} \right) - z w_{,s} \right] + \frac{w}{R}$$

$$\epsilon_s = u_{,s} + z \frac{u_{,s}}{R} - z \frac{u}{R^2} R_{,s} - z w_{,ss} + \frac{w}{R}$$

$$\epsilon_s = u_{,s} + \frac{w}{R} + z \left(\frac{u_{,s}}{R} - w_{,ss} - \frac{u}{R^2} R_{,s} \right)$$

Term not in Eq. 16.2-2 ↗

16.2-2

(a) $\epsilon_m = 0$ gives $u_{,s} = -\frac{w}{R}$ or $\frac{u_{,s}}{R} = -\frac{w}{R^2}$

$K=0$ gives $\frac{u_{,s}}{R} - w_{,ss} = 0$ or $w_{,ss} + \frac{w}{R^2} = 0$

Integrate; let $\phi = \frac{s}{R}$;

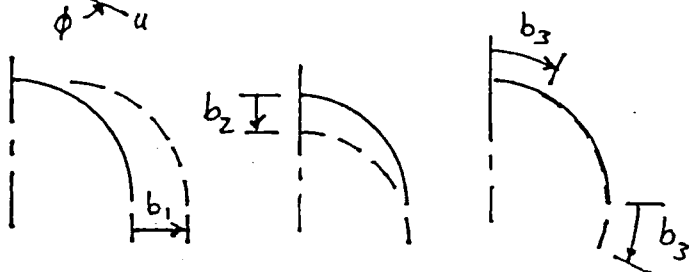
$w = b_1 \sin \phi - b_2 \cos \phi$; hence $u_{,s} = \frac{-b_1 \sin \phi + b_2 \cos \phi}{R}$

or $u_{,\phi} = -b_1 \sin \phi + b_2 \cos \phi$

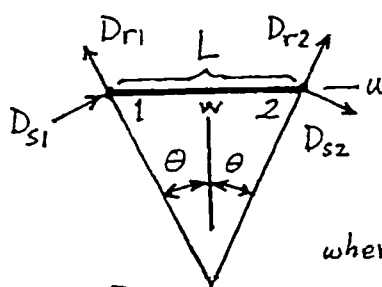
$u = b_1 \cos \phi + b_2 \sin \phi + b_3$

(b) $D_1 = u \cos \phi + w \sin \phi = b_1 + b_3 \cos \phi$

$D_2 = -u \sin \phi + w \cos \phi = -b_2 - b_3 \sin \phi$



16.2-3



$$\theta = \arcsin \frac{L/2}{R}$$

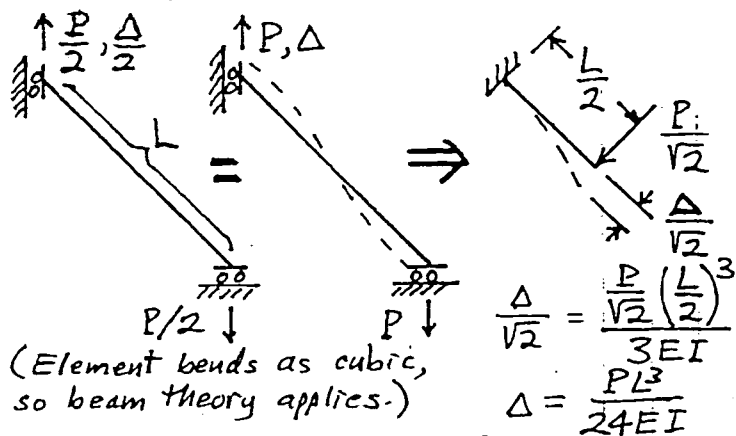
$$[k_{\text{global}}] = [T]^T [k_{\text{Eq. 16.2-6}}] [T]$$

where $[T]$ is

$$\begin{Bmatrix} u_1 \\ u_2 \\ w_1 \\ \psi_1 \\ w_2 \\ \psi_2 \end{Bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos \theta & \sin \theta & 0 \\ \sin \theta & \cos \theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} D_{s1} \\ D_{r1} \\ \psi_1 \\ D_{s2} \\ D_{r2} \\ \psi_2 \end{Bmatrix}$$

16.2-4

Exact: $\Delta = 0.1488 PR^3/EI$



- (a) $L = \sqrt{2}R$, $\Delta = 0.1179 PR^3/EI$ 20.8% low
- (b) $L = \frac{\pi R}{2}$, $\Delta = 0.1615 PR^3/EI$ 8.5% high

16.2-5

$$u = a_1 + a_2 s \quad \epsilon_m = a_2 + \frac{a_3 + a_4 s + a_5 s^2 + a_6 s^3}{R}$$

$$w = a_3 + a_4 s + a_5 s^2 + a_6 s^3 \quad \text{or}$$

$$\epsilon_m = b_1 + b_2 s + b_3 s^2 + b_4 s^3$$

where

$$b_1 = a_2 + \frac{a_3}{R}, \quad b_2 = \frac{a_4}{R}, \quad b_3 = \frac{a_5}{R}, \quad b_4 = \frac{a_6}{R}$$

$$\epsilon_m^2 = b_1^2 + b_2^2 s^2 + b_3^2 s^4 + b_4^2 s^6 + 2b_1 b_3 s^2 + 2b_2 b_4 s^4 + (\text{terms with odd powers of } s)$$

Odd powers will integrate to zero, so ignore.

Let $\beta_i = \text{constants}$; thus

$$\int_{-L/2}^{L/2} \epsilon_m^2 ds = \beta_1 b_1^2 + \beta_2 b_2^2 + \beta_3 b_3^2 + \beta_4 b_4^2 + \beta_5 b_1 b_3 + \beta_6 b_2 b_4$$

$$\int_{-L/2}^{L/2} \epsilon_m^2 ds = 0 \text{ implies } b_1 = b_2 = b_3 = b_4 = 0$$

$$\text{i.e. that } a_2 + \frac{a_3}{R} = a_4 = a_5 = a_6 = 0$$

16.2-6

Eqs. 16.2-9.

$$\Psi = w_{,s} = a_4 + 2a_5 s + 3a_6 s^2$$

node s
1 $-L/2$
2 $+L/2$

$$\begin{Bmatrix} u_1 \\ w_1 \\ \Psi_1 \\ u_2 \\ w_2 \\ \Psi_2 \end{Bmatrix} = \begin{bmatrix} 1 & -\frac{L}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{L}{2} & \frac{L^2}{4} & -\frac{L^3}{8} \\ 0 & 0 & 0 & 1 & -L & -\frac{3L^2}{2} \\ 1 & \frac{L}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{L}{2} & \frac{L^2}{4} & \frac{L^3}{8} \\ 0 & 0 & 0 & 1 & L & \frac{3L^2}{2} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

(b) Eqs. 16.2-12

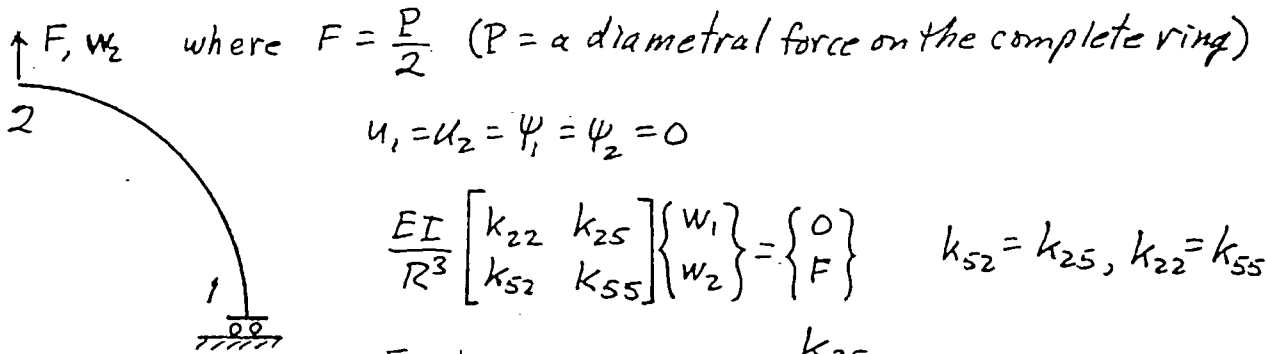
$$\Psi = w_{,s} = -\frac{1}{R}(2a_3 + 6a_4\phi + 12a_5\phi^2 + 20a_6\phi^3)$$

Node 1: $\phi = -L/2R$; Node 2: $\phi = +L/2R$

Let $\alpha = L/2R$

$$\begin{Bmatrix} u_1 \\ w_1 \\ \Psi_1 \\ u_2 \\ w_2 \\ \Psi_2 \end{Bmatrix} = \begin{bmatrix} 1 & -\alpha & \alpha^2 & -\alpha^3 & \alpha^4 & -\alpha^5 \\ 0 & -1 & 2\alpha & -3\alpha^2 & 4\alpha^3 & -5\alpha^4 \\ 0 & 0 & -\frac{2}{R} & \frac{6\alpha}{R} & -\frac{12\alpha^2}{R} & \frac{20\alpha^3}{R} \\ 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 & \alpha^5 \\ 0 & -1 & -2\alpha & -3\alpha^2 & -4\alpha^3 & -5\alpha^4 \\ 0 & 0 & -\frac{2}{R} & -\frac{6\alpha}{R} & -\frac{12\alpha^2}{R} & -\frac{20\alpha^3}{R} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

16.2-7



First eq. gives $w_1 = -\frac{k_{25}}{k_{22}} w_2$

Second eq. becomes

$$\left(-\frac{k_{25}^2}{k_{22}} + k_{55}\right) w_2 = F \frac{R^3}{EI}$$

$\beta = \frac{\pi R/2}{2R} = \frac{\pi}{4}$ hence $k_{22} = k_{25} = 42.9128$
 $k_{25} = 39.3948$

Therefore $w_2 = 0.1482 F \frac{R^3}{EI}$

Change in dia. of ring $= 2w_2 = 2 \left(0.1482 \frac{P}{2} \frac{R^3}{EI}\right) = 0.1482 \frac{PR^3}{EI}$
 (0.4% low)
 exact: $0.1488 \frac{PR^3}{EI}$

16.2-8

In elements of Eqs. 16.2-9, constraint $\epsilon_m = 0$ is not explicit; it is implicitly imposed (e.g. Eq. 16.2-11). If ϵ_m were ignored, membrane stiffness would be zero, not infinite ($\epsilon_m = 0$); singular $[k]$.

16.2-9

$$(a) \quad \epsilon_m = u_{1s} + \frac{w}{R} = a_2 + \frac{a_3 + a_4 s}{R}$$

$$r_{2s} = w_{1s} - \beta = a_4 - (a_5 + a_6 s)$$

$\epsilon_m = 0$ for all s implies

$$\left. \begin{array}{l} a_2 + \frac{a_3}{R} = a_4 = 0 \\ a_4 - a_5 = a_6 = 0 \end{array} \right\} \therefore a_5 = 0$$

For a straight el., $R \rightarrow \infty$, and

$$a_2 = a_4 - a_5 = a_6 = 0$$

(b) Evaluate at $s=0$:

$$\left. \begin{array}{l} \epsilon_m = a_2 + \frac{a_3}{R} \\ r_{2s} = a_4 - a_5 \end{array} \right\} \text{constraints } a_2 + \frac{a_3}{R} = a_4 - a_5 = 0$$

16.2-10

$$\gamma_{zs} = w_{is} - \beta = \frac{2a_5}{L} + \frac{4a_6}{L}\xi - a_7 - a_8\xi - a_9\xi^2$$

$$\gamma_{zs} = \left(\frac{2a_5}{L} - a_7\right) + \left(\frac{4a_6}{L} - a_8\right)\xi - (a_9)\xi^2 \quad (A)$$

$\gamma_{zs} = 0$ for all ξ implies that all three parenthetical expressions in Eq. (A) vanish.

Note; $a_9 = 0$ implies $\beta_{ss} = 0$.

Rewrite Eq. (A):

$$0 = \left(\frac{2a_5}{L} - a_7 - \frac{a_9}{3}\right) + \left(\frac{4a_6}{L} - a_8\right)\xi - \left[\xi^2 - \frac{1}{3}\right]a_9$$

At Gauss pts, $\xi = \pm \frac{1}{\sqrt{3}}$, $\left[\xi^2 - \frac{1}{3}\right] = 0$.

Constraints are then $\frac{2a_5}{L} - a_7 - \frac{a_9}{3} = 0$
 $\frac{4a_6}{L} - a_8 = 0$

16.3-1

$$U_m = \frac{1}{2} \int_{-L/2}^{L/2} \frac{Et}{1-\nu^2} \begin{Bmatrix} \epsilon_{ms} \\ \epsilon_{m\theta} \end{Bmatrix}^T \begin{bmatrix} 1 & -\nu \\ -\nu & 1 \end{bmatrix} \begin{Bmatrix} \epsilon_{ms} \\ \epsilon_{m\theta} \end{Bmatrix} 2\pi R ds$$

Unrestrained axially: $\epsilon_{ms} = -\nu \epsilon_{m\theta} = -\nu \frac{w}{R}$

$$U_m = \frac{1}{2} \int_{-L/2}^{L/2} \frac{Et}{1-\nu^2} \frac{w^2}{R^2} \begin{Bmatrix} -\nu \\ 1 \end{Bmatrix}^T \begin{bmatrix} 1 & -\nu \\ -\nu & 1 \end{bmatrix} \begin{Bmatrix} -\nu \\ 1 \end{Bmatrix} 2\pi R ds$$

$$U_m = \frac{1}{2} \int_{-L/2}^{L/2} \frac{Et}{1-\nu^2} \frac{w^2}{R^2} (1-\nu^2) 2\pi R ds = \frac{1}{2} \int \frac{Et}{R^2} w^2 dA$$

Compare with Eq. 8.8-1:

foundation modulus is $\beta = \frac{Et}{R^2}$

16.3-2

$$U = \frac{1}{2} \int_{-L/2}^{L/2} E \epsilon_{m\theta}^2 (2\pi R t) ds + \frac{1}{2} \int_{-L/2}^{L/2} D \kappa_s^2 (2\pi R) ds \quad (\text{see Fig. 16.3-1b})$$

where, from Eq. 16.3-3, $\epsilon_{m\theta} = \frac{w}{R}$, $\kappa_s = \frac{d^2 w}{ds^2}$. Also $D = \frac{Et^3}{12(1-\nu^2)}$

$$[k] = [k_m] + [k_b] = \frac{2\pi Et}{R} \int_{-L/2}^{L/2} [N]^T [N] ds + 2\pi DR \int_{-L/2}^{L/2} [N_{,ss}]^T [N_{,ss}] ds$$

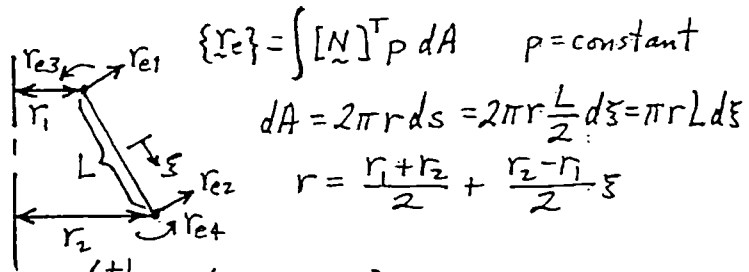
$[k_m]$ has the form of a mass matrix (Eq. 11.2-6) with $2\pi Et/R$ in place of m ; thus

$$[k_m] = \frac{2\pi EtL}{420R} \begin{bmatrix} 156 & 22L & 54 & -13L \\ & 4L^2 & 13L & -3L^2 \\ & & 156 & -22L \\ \text{symm.} & & & 4L^2 \end{bmatrix}$$

$[k_b]$ has the form of a standard beam element matrix with $2\pi RD$ in place of EI ; thus

$$[k_b] = \frac{2\pi RD}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ \text{symm.} & & & 4L^2 \end{bmatrix}$$

16.3-3



$\{r_e\} = [N]^T p dA \quad p = \text{constant}$
 $dA = 2\pi r ds = 2\pi r \frac{L}{2} d\xi = \pi r L d\xi$
 $r = \frac{r_1 + r_2}{2} + \frac{r_2 - r_1}{2} \xi$

$$\{r_e\} = \int_{-1}^{+1} \frac{p\pi L}{2} \begin{Bmatrix} (1-\xi)/2 \\ (1+\xi)/2 \\ (1-\xi^2)L/8 \\ -(1-\xi^2)L/8 \end{Bmatrix} [(r_1 + r_2) + (r_2 - r_1)\xi] d\xi$$

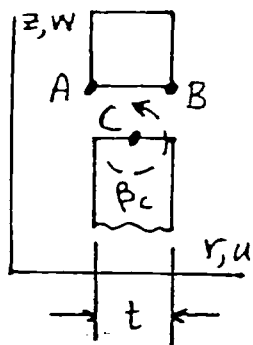
$$\{r_e\} = \frac{p\pi L}{2} (r_1 + r_2) \begin{Bmatrix} 1 \\ 1 \\ \frac{L}{8} \left(\frac{4}{3} \right) \\ -\frac{L}{8} \left(\frac{4}{3} \right) \end{Bmatrix} + \frac{p\pi L}{2} (r_2 - r_1) \begin{Bmatrix} -\frac{1}{3} \\ +\frac{1}{3} \\ 0 \\ 0 \end{Bmatrix}$$

$$\{r_e\} = \frac{p\pi L}{2} (r_1 + r_2) \begin{Bmatrix} 1 \\ 1 \\ L/6 \\ -L/6 \end{Bmatrix} + \frac{p\pi L (r_2 - r_1)}{6} \begin{Bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$$

16.3-4

(a)

Axially symmetric:



$$\begin{Bmatrix} w_c \\ u_c \\ \beta_c \end{Bmatrix} = [T] \begin{Bmatrix} w_A \\ u_A \\ w_B \\ u_B \end{Bmatrix}$$

3×4

where, following arguments in Section 8.5, $[T]$ is

$$\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{t} & 0 & \frac{1}{t} & 0 \end{bmatrix}$$

(b) Circumferential displacement v enters.

$$\begin{Bmatrix} w_c \\ u_c \\ v_c \\ \beta_c \end{Bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ -\frac{1}{t} & 0 & 0 & \frac{1}{t} & 0 & 0 \end{bmatrix} \begin{Bmatrix} w_A \\ u_A \\ v_A \\ w_B \\ u_B \\ v_B \end{Bmatrix}$$

(c) For the case without axial symmetry, we write

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & -1 & 0 \\ -\frac{1}{t} & 0 & 0 & \frac{1}{t} & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} w_A \\ u_A \\ v_A \\ w_B \\ u_B \\ v_B \\ w_c \\ u_c \\ v_c \\ \beta_c \end{Bmatrix} = \{0\}$$

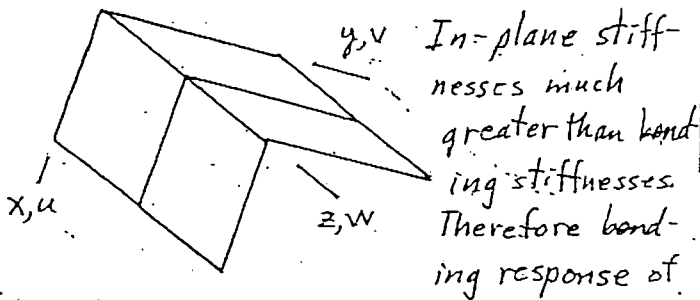
To account for other d.o.f. in the structure above & below section mc , expand the rectangular matrix left & right by adding zeros, & expand the column vector up & down by adding d.o.f.

If there is axial symmetry, we can omit the information content associated with v_A , v_B , and v_c .

16.4-1

Let s be a coordinate parallel to an element edge. Element-normal displacement is cubic or quadratic in s (depending on element type), while element-tangent (but edge-normal) displacement is linear in s . Thus there is incompatibility, most strongly if adjacent elements meet at a right angle. There is no incompatibility if elements are coplanar, and such a condition is approached as the mesh is indefinitely refined.

16.4-2



In-plane stiffnesses much greater than bending stiffnesses. Therefore bonding response of each side is modeled well if nodal d.o.f. Ψ_y and Ψ_x set to zero (retain Ψ_z). Indeed, each side could be analyzed separately, except that Ψ_z provides elastic support to element edges that lie on the z axis.

16.5-1

Let j and k refer to upper and lower surface nodes respectively. The N 's are given by Eqs. 6.4-1.

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum_{k=1}^8 N_k \frac{1-\xi}{2} \begin{Bmatrix} x_k \\ y_k \\ z_k \end{Bmatrix} + \sum_{j=1}^8 N_j \frac{1+\xi}{2} \begin{Bmatrix} x_j \\ y_j \\ z_j \end{Bmatrix}$$

(a) $[J] = \begin{bmatrix} x_{,5} & y_{,5} & z_{,5} \\ x_{,7} & y_{,7} & z_{,7} \\ x_{,5} & y_{,5} & z_{,5} \end{bmatrix}$ From Eq. 16.5-2,

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} N_i & 0 & 0 & N_i \xi/2 & 0 & 0 \\ 0 & N_i & 0 & 0 & N_i \eta/2 & 0 \\ 0 & 0 & N_i & 0 & 0 & N_i \xi/2 \end{bmatrix} \begin{Bmatrix} x_i \\ y_i \\ z_i \\ t_i/3i \\ t_i m_{3i} \\ t_i n_{3i} \end{Bmatrix}$$

Terms in (e.g.) col. 1 of $[J]$ are

$$\frac{\partial x}{\partial \xi} = \sum \begin{bmatrix} N_{i,\xi} & 0 & 0 & \frac{\xi}{2} N_{i,\xi} & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_i \\ y_i \\ z_i \\ t_i/3i \\ t_i m_{3i} \\ t_i n_{3i} \end{Bmatrix}_{6 \times 1}$$

$$\frac{\partial x}{\partial \eta} = \sum \begin{bmatrix} N_{i,\eta} & 0 & 0 & \frac{\eta}{2} N_{i,\eta} & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_i \\ y_i \\ z_i \\ t_i/3i \\ t_i m_{3i} \\ t_i n_{3i} \end{Bmatrix}_{6 \times 1}$$

$$\frac{\partial x}{\partial \xi} = \sum \begin{bmatrix} 0 & 0 & 0 & \frac{N_i}{2} & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_i \\ y_i \\ z_i \\ t_i/3i \\ t_i m_{3i} \\ t_i n_{3i} \end{Bmatrix}_{6 \times 1}$$

(the N_i are functions of ξ and η but not of ξ). Sum over no. of nodes in element.

(b) Here $z_i = 1, m_{3i} = 0, n_{3i} = 1, t_i = t$.

Note that $\sum N_i = 1$ and $\sum N_{i,\xi} = \sum N_{i,\eta} = \sum N_{i,\xi} = 0$.

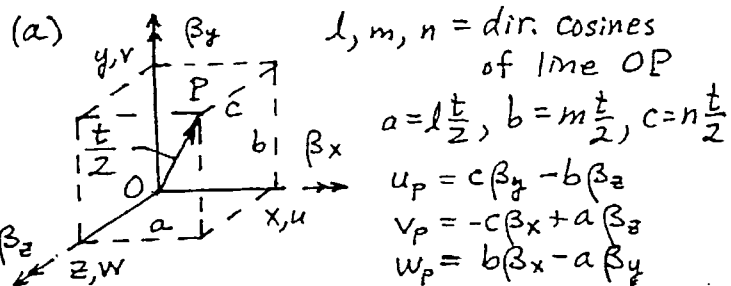
$$\frac{\partial x}{\partial \xi} = \sum N_{i,\xi} x_i \quad \frac{\partial y}{\partial \xi} = \sum N_{i,\xi} y_i \quad \frac{\partial z}{\partial \xi} = 0$$

$$\frac{\partial x}{\partial \eta} = \sum N_{i,\eta} x_i \quad \frac{\partial y}{\partial \eta} = \sum N_{i,\eta} y_i \quad \frac{\partial z}{\partial \eta} = 0$$

$$\frac{\partial x}{\partial \xi} = 0 \quad \frac{\partial y}{\partial \xi} = 0 \quad \frac{\partial z}{\partial \xi} = \frac{t}{2}$$

Thus $[J]$ becomes as for a flat plate.

16.5-3



$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum N_i \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} + \sum N_i \frac{t_i}{2} \begin{bmatrix} 0 & n_i & -m_i \\ -n_i & 0 & l_i \\ m_i & -l_i & 0 \end{bmatrix} \begin{Bmatrix} \beta_x \\ \beta_y \\ \beta_z \end{Bmatrix}$$

(b) E.g., if V_3 is x-parallel, then $l_i = 1$, $m_i = n_i = 0$, β_y becomes α and β_z becomes β (see Fig. 16.5-2). And, in Fig. 16.5-2b, V_1 becomes y-parallel & V_2 becomes z-parallel. Thus, $[u][\beta]$ products in above eq. & in Eq. 16.5-6 become respectively

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{Bmatrix} \beta_x \\ \alpha \\ \beta \end{Bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} \quad \text{These agree.}$$

16.5-4

$$\begin{Bmatrix} \epsilon_x \\ \vdots \\ \delta_{2x} \end{Bmatrix} = [H] \begin{Bmatrix} u_x \\ \vdots \\ w_z \end{Bmatrix} = [H] \begin{bmatrix} \Gamma & 0 & 0 \\ 0 & \Gamma & 0 \\ 0 & 0 & \Gamma \end{bmatrix} \begin{Bmatrix} u_s \\ \vdots \\ w_s \end{Bmatrix}$$

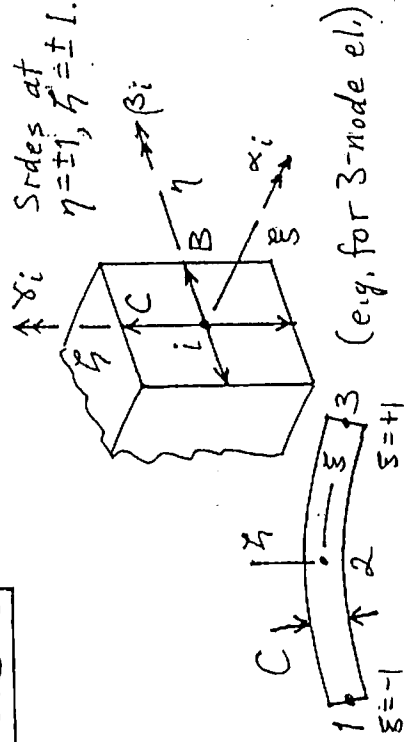
where $[\tilde{J}] = [\tilde{\Gamma}]^{-1}$ is given by Eq. 6.5-2,

$[H]$ is given by Eq. 6.5-3, and the column vector $[u_s \dots w_s]^T$ by Eq. 16.5-9. We want

$$[\tilde{B}_i] = \begin{bmatrix} \Gamma & 0 & 0 \\ 0 & \Gamma & 0 \\ 0 & 0 & \Gamma \end{bmatrix} \text{ [square array in Eq. 16.5-9]}$$

Let $A_i = \Gamma_{11} N_{i,s} + \Gamma_{12} N_{i,\eta}$ $B_i = \Gamma_{21} N_{i,s} + \Gamma_{22} N_{i,\eta}$ $C_i = \Gamma_{31} N_{i,s} + \Gamma_{32} N_{i,\eta}$. Then $[\tilde{B}_i] =$

$$\begin{bmatrix} A_i & 0 & 0 & -(A_i \zeta + \Gamma_{13} N_i) l_{2i} t_i / 2 & (A_i \zeta + \Gamma_{13} N_i) l_{1i} t_i / 2 \\ B_i & 0 & 0 & -(B_i \zeta + \Gamma_{23} N_i) l_{2i} t_i / 2 & (B_i \zeta + \Gamma_{23} N_i) l_{1i} t_i / 2 \\ C_i & 0 & 0 & -(C_i \zeta + \Gamma_{33} N_i) l_{2i} t_i / 2 & (C_i \zeta + \Gamma_{33} N_i) l_{1i} t_i / 2 \\ 0 & A_i & 0 & -(A_i \zeta + \Gamma_{13} N_i) m_{2i} t_i / 2 & (A_i \zeta + \Gamma_{13} N_i) m_{1i} t_i / 2 \\ 0 & B_i & 0 & -(B_i \zeta + \Gamma_{23} N_i) m_{2i} t_i / 2 & (B_i \zeta + \Gamma_{23} N_i) m_{1i} t_i / 2 \\ 0 & C_i & 0 & -(C_i \zeta + \Gamma_{33} N_i) m_{2i} t_i / 2 & (C_i \zeta + \Gamma_{33} N_i) m_{1i} t_i / 2 \\ 0 & 0 & A_i & -(A_i \zeta + \Gamma_{13} N_i) n_{2i} t_i / 2 & (A_i \zeta + \Gamma_{13} N_i) n_{1i} t_i / 2 \\ 0 & 0 & B_i & -(B_i \zeta + \Gamma_{23} N_i) n_{2i} t_i / 2 & (B_i \zeta + \Gamma_{23} N_i) n_{1i} t_i / 2 \\ 0 & 0 & C_i & -(C_i \zeta + \Gamma_{33} N_i) n_{2i} t_i / 2 & (C_i \zeta + \Gamma_{33} N_i) n_{1i} t_i / 2 \end{bmatrix}$$



$$(a) \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum N_i \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} + \sum \frac{B}{2} \eta N_i \begin{Bmatrix} l_{2i} \\ m_{2i} \\ n_{2i} \end{Bmatrix} + \sum \frac{C}{2} \xi N_i \begin{Bmatrix} l_{3i} \\ m_{3i} \\ n_{3i} \end{Bmatrix}$$

$$(b) \begin{aligned} |V_{2i}| &= B; \\ |V_{3i}| &= C \end{aligned}$$

Consider e.g. motion of corner $\eta = \xi = +1$; due to rotations, it is:

$$\begin{aligned} &\beta_i C/2 - \gamma_i B/2 \quad \text{in } \xi \text{ direction} \\ &-\alpha_i C/2 \quad \text{in } \eta \text{ direction} \\ &\alpha_i B/2 \quad \text{in } \xi \text{ direction} \end{aligned}$$

In this way, and resolving into x, y, z components, for an arbitrary point,

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum N_i \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix}$$

$$\begin{aligned} &\begin{bmatrix} l_{3i} & 0 & -l_{1i} \\ m_{3i} & 0 & -m_{1i} \\ n_{3i} & 0 & -n_{1i} \end{bmatrix} \begin{Bmatrix} \alpha_i \\ \beta_i \\ \gamma_i \end{Bmatrix} \\ &+ \sum N_i \xi \frac{C}{2} \begin{bmatrix} -l_{2i} & l_{1i} & 0 \\ -m_{2i} & m_{1i} & 0 \\ -n_{2i} & n_{1i} & 0 \end{bmatrix} \begin{Bmatrix} \alpha_i \\ \beta_i \\ \gamma_i \end{Bmatrix} \end{aligned}$$

17.2-1

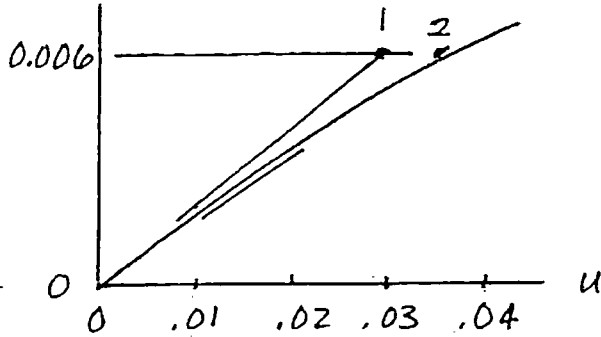
(a) $(0.2-u)u = 0.006$; $u_{\text{exact}} = 0.0367544$

Direct substitution: $(0.2-0)u_1 = 0.006$, $u_1 = 0.030$

$(0.2-0.030)u_2 = 0.006$, $u_2 = 0.03529$

$(0.2-0.03529)u_3 = 0.006$, $u_3 = 0.03643$

(b) P



Step 0-1 is tangent to curve at $u=0$

(c) $(0.2-u)u = P$

$P = 0.2u - u^2$

$k_t = \frac{dP}{du} = 0.2 - 2u$

17.2-2

$$(a) \frac{10}{(D+1)^2} \Delta D = P - R_c = 8 - R_c$$

$$\frac{10}{(D_i+1)^2} \Delta D_{i+1} = 8 - \frac{10D_i}{D_i+1} \quad \uparrow \text{resisting force}$$

$$D_{i+1} = D_i + \Delta D_{i+1}$$

Start with $D=0$

when $i=0$. Thus

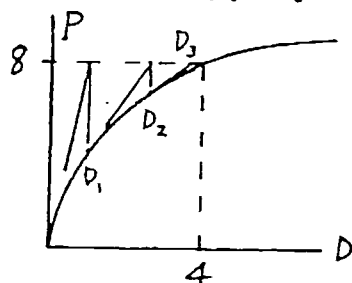
$$D_1 = 0.800$$

$$D_2 = 1.952$$

$$D_3 = 3.161$$

$$D_4 = 3.859$$

$$D_5 = 3.996$$



$$(b) \text{ At } D=3, k_t = \frac{dP}{dD} = \frac{10}{(3+1)^2} = 0.625$$

$$0.625 \Delta D = P - R_c, \quad 0.625 \Delta D_{i+1} = 8 - \frac{10D_i}{D_i+1}$$

and $D_{i+1} = D_i + \Delta D_{i+1}$. Start with $D=3$ at $i=0$.

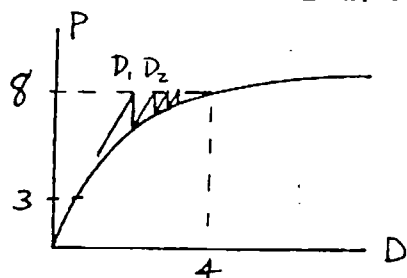
$$D_1 = 3.800$$

$$D_2 = 3.933$$

$$D_3 = 3.977$$

$$D_4 = 3.992$$

$$D_5 = 3.997$$



(c) Obtain $D_1 = 3.8$ as in part (b). Then

$$k_t = \frac{10}{(3.8+1)^2} = 0.43403. \text{ Then}$$

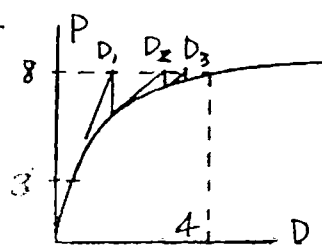
$$0.43403 \Delta D_{i+1} = 8 - \frac{10D_i}{D_i+1}$$

We obtain

$$D_2 = 3.992$$

$$D_3 = 3.99995$$

$$D_4 = 3.99995$$



$$(d) \frac{10}{(D_i+1)^2} \Delta D_{i+1} = \Delta P = 2, \quad D_{i+1} = D_i + \Delta D_{i+1}$$

Start with $P=1$ at $D=\frac{1}{9}$.

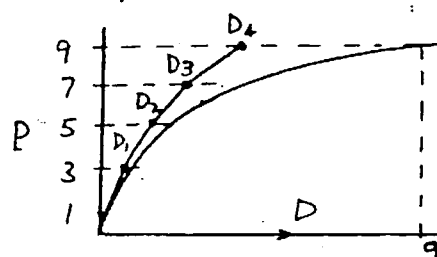
$$P_0 = 1 \quad D_0 = \frac{1}{9}$$

$$P_1 = 3 \quad D_1 = 0.358$$

$$P_2 = 5 \quad D_2 = 0.727$$

$$P_3 = 7 \quad D_3 = 1.323$$

$$P_4 = 9 \quad D_4 = 2.403$$



(e) After calc. of $D_1 = 0.358$ in part (d), write

$$\frac{10}{(D_i+1)^2} \Delta D_{i+1} = 2 + \left(P_i - \frac{10}{D_i+1} \right)$$

where P_i is the load applied to get D_i , not the load associated with $D_{i+1} = D_i + \Delta D_{i+1}$.

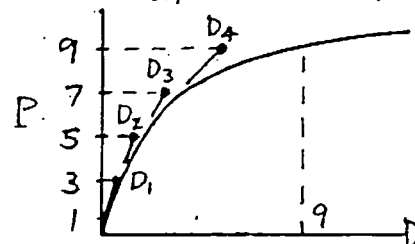
$$P_0 = 1 \quad D_0 = \frac{1}{9}$$

$$P_1 = 3 \quad D_1 = 0.358$$

$$P_2 = 5 \quad D_2 = 0.794$$

$$P_3 = 7 \quad D_3 = 1.622$$

$$P_4 = 9 \quad D_4 = 3.557$$



$$(f) \frac{10}{D_i+1} D_{i+1} = R$$

$$D_{i+1} = \frac{D_i+1}{10} R = 0.8(D_i+1). \text{ Start with } D_0=0.$$

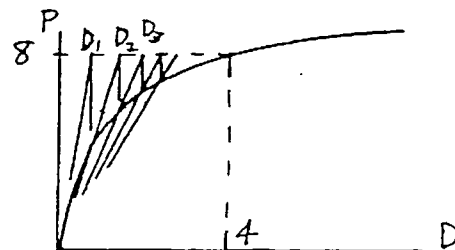
$$D_1 = 0.800$$

$$D_2 = 1.440$$

$$D_3 = 1.952$$

$$D_4 = 2.362$$

$$D_5 = 2.689$$

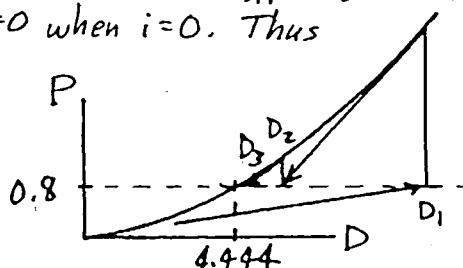


17.2-3

(a) $\frac{10}{(10-D)^2} \Delta D = P - R_c = 0.8 - R_c$
 $\frac{10}{(10-D_i)^2} \Delta D_{i+1} = 0.8 - \frac{D_i}{10-D_i}$ $\nearrow$ resisting force
 $D_{i+1} = D_i + \Delta D_{i+1}$

Start with $D=0$ when $i=0$. Thus

$D_1 = 8.000$
 $D_2 = 6.720$
 $D_3 = 5.377$
 $D_4 = 4.601$
 $D_5 = 4.449$

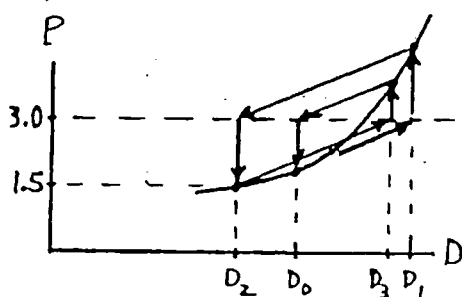


(b) At $D=6$, $k_t = \frac{dP}{dD} = \frac{10}{(10-6)^2} = 0.625$

$0.625 \Delta D = P - R_c$, $0.625 \Delta D_{i+1} = 3 - \frac{D_i}{10-D_i}$

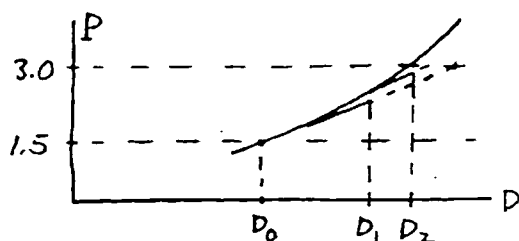
and $D_{i+1} = D_i + \Delta D_{i+1}$. Start with $D=6$ at $i=0$.

$D_1 = 8.400$
 $D_2 = 4.800$
 $D_3 = 8.123$
 $D_4 = 5.998$
 $D_5 = 8.400$



We are stuck in a loop.

(c) i	D_i	ΔD_{i+1}	$0.6 \Delta D_{i+1}$	D_{i+1}
0	6.000	2.400	1.440	7.440
1	7.440	0.150	0.090	7.530
2	7.530	-0.078	-0.047	7.483
3	7.483	0.042	0.025	7.509
4	7.509	-0.023	-0.014	7.495



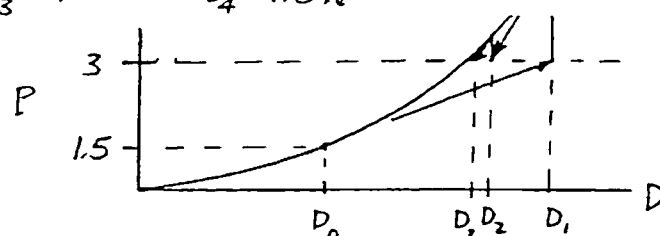
Exact result at $P=3$ is $D=7.50$.

(d) Obtain $D_1 = 8.4$ as in Part (b). Then

$k_t = \frac{10}{(10-8.4)^2} = 3.90625$, Then

$3.90625 \Delta D_{i+1} = 3 - \frac{D_i}{10-D_i}$, $D_{i+1} = D_i + \Delta D_{i+1}$

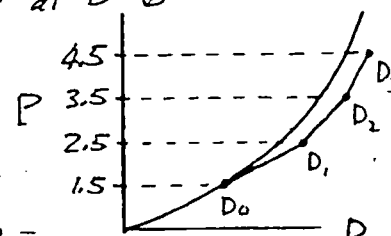
$D_1 = 8.400$ $D_2 = 7.824$
 $D_3 = 7.672$ $D_4 = 7.596$



(e) $\frac{10}{(10-D_i)^2} \Delta D_{i+1} = \Delta P = 1$, $D_{i+1} = D_i + \Delta D_{i+1}$

Start with $P=1.5$ at $D=6$

$P_0 = 1.5$, $D_0 = 6.000$
 $P_1 = 2.5$, $D_1 = 7.600$
 $P_2 = 3.5$, $D_2 = 8.176$
 $P_3 = 4.5$, $D_3 = 8.509$



(f) After calc. of $D_1 =$

7.600 in Part (e), write

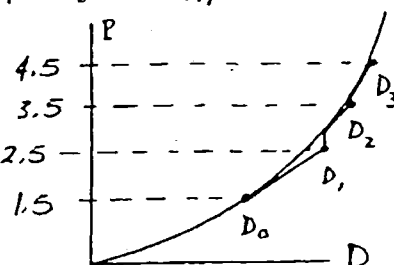
$\frac{10}{(10-D_i)^2} \Delta D_{i+1} = 1 + \left(P_i - \frac{D_i}{10-D_i} \right)$, where P_i is the load applied to obtain D_i , not the load associated with $D_{i+1} = D_i + \Delta D_{i+1}$.

$P_0 = 1.5$, $D_0 = 6.000$

$P_1 = 2.5$, $D_1 = 7.600$

$P_2 = 3.5$, $D_2 = 7.792$

$P_3 = 4.5$, $D_3 = 8.265$

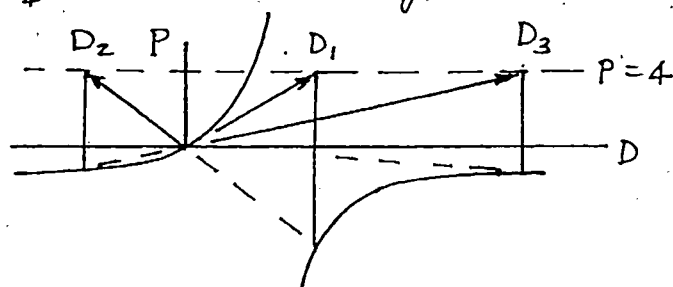


(g) $(10-D)P = D$

$D_{i+1} = P(10-D_i) = 4(10-D_i)$

$D_0 = 0$, $D_1 = 40$, $D_2 = -120$, $D_3 = 520$,

$D_4 = -2040$: diverges.



17.2-4

(a) Let L = length of bar.

$$L = [a^2 + (c-D)^2]^{1/2} = a \left[1 + \left(\frac{c-D}{a} \right)^2 \right]^{1/2}$$

$$L \approx a \left[1 + \frac{1}{2} \left(\frac{c-D}{a} \right)^2 \right] \text{ for } c-D \ll a$$

$$L_0 \approx a \left[1 + \frac{1}{2} \left(\frac{c}{a} \right)^2 \right] \text{ for } D=0.$$

$$\epsilon = \frac{L-L_0}{L_0} \approx \frac{L-L_0}{a} \approx \frac{1}{2} \left(\frac{c-D}{a} \right)^2 - \frac{1}{2} \frac{c^2}{a^2} = \frac{-2cD+D^2}{2a^2}$$

$$\Pi_P = \int_0^L \frac{E}{2} \epsilon^2 A dx - PD \approx \frac{AE}{8a^3} (-2cD+D^2)^2 - PD$$

$$(b) 0 = \frac{\partial \Pi_P}{\partial D} = \frac{AE}{4a^3} (-2cD+D^2)(-2c+2D) - P$$

$$\text{or } 0 = \frac{AE}{2a^3} (2cD-D^2)(c-D) - P \quad (A)$$

(c) Eq. (A) above is

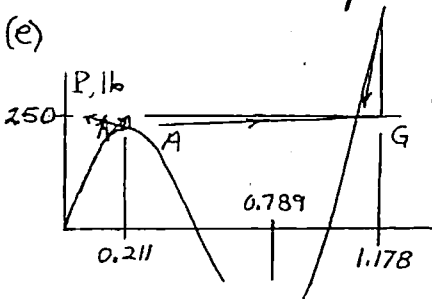
$$\left[\frac{AE}{2a^3} (2c^2 - 3cD + D^2) \right] D = P, \text{ i.e. } K_{\text{secant}} D = P$$

$$\frac{dP}{dD} = \frac{AE}{2a^3} (2c^2 - 6cD + 3D^2) = K_{\text{tangent}}$$

(d) Limit points, where $K_{\text{tangent}} = 0$

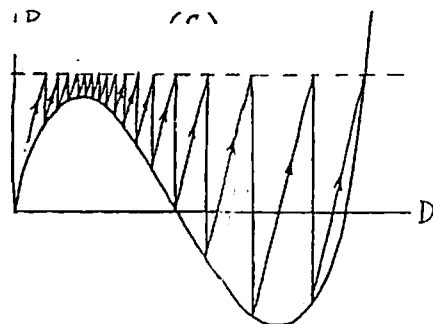
$$2c^2 - 6cD + 3D^2 = 0 \text{ yields } D = c \left(1 \pm \frac{\sqrt{3}}{3} \right)$$

(e)



The solution dances about on top of the first hill, unable to find a D for which the resistance of the structure is equal to the load.

Only when we move along AG, which has a small enough slope that we hit $P=250$ at a D greater than 0.789, can we converge.



17.2-5

$$P_1 = f_1(D_1, D_2) \quad P_2 = f_2(D_1, D_2)$$

$$P_A = f_1(D_A, D_B) = f_1(D_A^* + \Delta D_A, D_B^* + \Delta D_B)$$

$$P_B = f_2(D_A, D_B) = f_2(D_A^* + \Delta D_A, D_B^* + \Delta D_B)$$

Apply truncated Taylor series.

$$P_A = f_1(D_A^*, D_B^*) + \Delta D_A \left. \frac{\partial f_1}{\partial D_1} \right|_{D_A^*, D_B^*} + \Delta D_B \left. \frac{\partial f_1}{\partial D_2} \right|_{D_A^*, D_B^*}$$

$$P_B = f_2(D_A^*, D_B^*) + \Delta D_A \left. \frac{\partial f_2}{\partial D_1} \right|_{D_A^*, D_B^*} + \Delta D_B \left. \frac{\partial f_2}{\partial D_2} \right|_{D_A^*, D_B^*}$$

Group terms

$$\begin{bmatrix} \partial f_1 / \partial D_1 & \partial f_1 / \partial D_2 \\ \partial f_2 / \partial D_1 & \partial f_2 / \partial D_2 \end{bmatrix}_{D_A^*, D_B^*} \begin{Bmatrix} \Delta D_A \\ \Delta D_B \end{Bmatrix} = \begin{Bmatrix} P_A - f_1(D_A^*, D_B^*) \\ P_B - f_2(D_A^*, D_B^*) \end{Bmatrix}$$

$\underbrace{\begin{bmatrix} \partial f_1 / \partial D_1 & \partial f_1 / \partial D_2 \\ \partial f_2 / \partial D_1 & \partial f_2 / \partial D_2 \end{bmatrix}_{D_A^*, D_B^*}}_{\text{tangent stiffness at } D_A^*, D_B^*} \begin{Bmatrix} \Delta D_A \\ \Delta D_B \end{Bmatrix} = \underbrace{\begin{Bmatrix} P_A - f_1(D_A^*, D_B^*) \\ P_B - f_2(D_A^*, D_B^*) \end{Bmatrix}}_{\text{load imbalance disp. increments}}$

17.2-6

$$g'(x^*) = 1 - \frac{f'(x^*)^2 - f''(x^*)f(x^*)}{f'(x^*)^2}$$

$$g'(x^*) = 1 - 1 + \frac{f''(x^*)f(x^*)}{f'(x^*)^2}$$

But $f(x^*) = 0$, so $g'(x^*) = 0$. Also $g(x^*) = x^*$

Hence

$$x_{i+1} = g(x_i) = \underbrace{g(x^*)}_{=x^*} + \underbrace{g'(x^*)(x_i - x^*)}_{=0} + \frac{1}{2} g''(\bar{x}_i)(x_i - x^*)^2$$

Thus

$$|x_{i+1} - x^*| = \frac{1}{2} |g''(\bar{x}_i)| (x_i - x^*)^2$$

$$\text{or } e_{i+1} = C e_i^2$$

17.3-1

At start, $K_t = 2 \frac{AE}{L} = 2 \frac{(1)10,000}{10} = 2000$

$2000 \Delta D = 20$, $\Delta D = 0.01$, $D = 0 + \Delta D = 0.01$

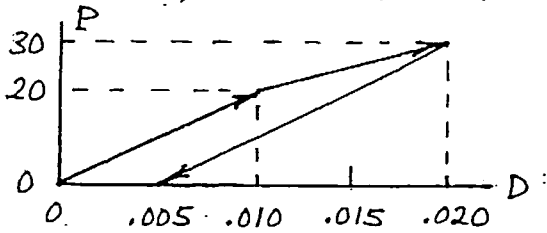
$\Delta \epsilon = 0.001$, $\epsilon = \pm 0.001$. Now $E_t = 0$ in left bar.

$K_t = \frac{AE}{L} = 1000$, $1000 \Delta D = 1000$, $\Delta D = 0.01$,

$D = 0.01 + \Delta D = 0.02$. Now unload (elastically)

$K_t = 2000$ again; $2000 \Delta D = -30$,

$\Delta D = -0.015$, $D = 0.02 + (-0.015) = +0.005$



17.3-2

To yield point: $K = 2(AE/L) = 2000$,
 $KD = P$, where $P = 20$ & $D = 0.01$. Sub-
 sequent strain increments are entirely plas-
 tic in left el., $\Delta\epsilon^P = \Delta\epsilon$. For $\Delta P = 10$,
 $2000 \Delta D = 10$, $\Delta D = 0.005$, $D = 0.01 + \Delta D = 0.015$
 Load ΔR , is $AE \Delta\epsilon^P = (1)10,000 \Delta\epsilon^P$. Hence
 $2000 \Delta D_i = 10,000 \Delta\epsilon^P = 10,000 (\Delta D_{i-1} / 10)$

i.e. $\Delta D_i = 0.5 \Delta D_{i-1}$

& $D_i = D_{i-1} + \Delta D_{i-1}$

ΔD	D
0.005	0.015
0.0025	0.0175
0.00125	0.01875
0.000625	0.019375
0.0003125	0.0196875
⋮	⋮
0.0	0.020

Successive iterates
 are as shown at
 right.

17.3-3

For each bar, $k = AE/L = 0.5$ until yielding; $k = 0$ after yielding. Before yielding, force in each bar is $F = 0.5v$.

1st Step $K = 3k = 1.5$. Let $\Delta P = 1$.

$$1.5 \Delta v = 1, \Delta v = \frac{2}{3}, v = 0 + \Delta v = \frac{2}{3}$$

Scale solution by c to make $F_1 = 2$:

$$F_1 = 2 = 0.5 \left(\frac{2}{3} \right) c, c = 6, \text{ so } v = 6 \left(\frac{2}{3} \right) = 4$$

2nd Step $K = 2k = 1$. Let $\Delta P = 1$.

(1) $\Delta v = 1, \Delta v = 1$. Scale to make $F_2 = 4$:

$$0.5(4 + c \Delta v) = F_2 = 4, 4 + c = 8, c = 4$$

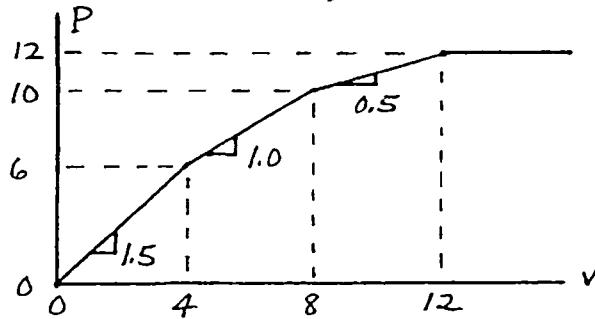
$$\text{Now } v = 4 + 4(1) = 8, P = 6 + 4(1) = 10$$

3rd Step $K = (1)k = 0.5$. Let $\Delta P = 1$.

$0.5 \Delta v = 1, \Delta v = 2$. Scale to make $F_3 = 6$:

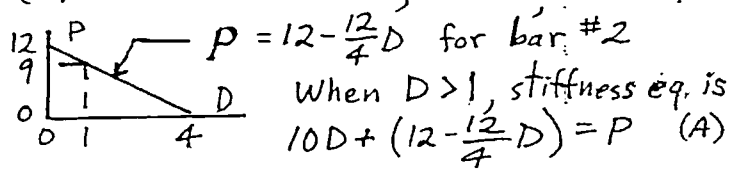
$$0.5(8 + c \Delta v) = F_3 = 6, 4 + c = 6, c = 2$$

$$\text{Now } v = 8 + 2(2) = 12, P = 10 + 2(1) = 12$$



17.3-4

(a) When $D < 1$, $P < 19$, $K = 19$



For $P = 24$, $D = 12/7 = 1.714286$

(b) When $D = 1$, $P = 10 + 9 = 19$

Load left to apply is $24 - 19 = 5$

From Eq. (A), for $D > 1$, $K = 10 - \frac{12}{4} = 7$

Hence $K \Delta D = \Delta P$ is $7 \Delta D = 5$ and

$\Delta D = \frac{5}{7} = 0.714286$, $D = 1 + \Delta D = 1.714286$

To start, assume standard linearly elastic structure. Form $[K]$; apply a reference load; calculate $\{D\}$ and bar forces. For each bar, calculate ratio of actual load to yield point load (or buckling load for compression members). Select the largest of these ratios and scale the entire solution by dividing by it. We are now at the end of the linear regime.

Now remove the bar about to yield or buckle. Load the structure nodes to which it was attached by the yield or buckling force in the bar removed. Form $[K]$ for the remaining structure; apply load increment; calculate $\{\Delta D\}$ and increments in bar forces. Determine r_i , the load increment ratio needed to cause each remaining bar to fail, i.e.

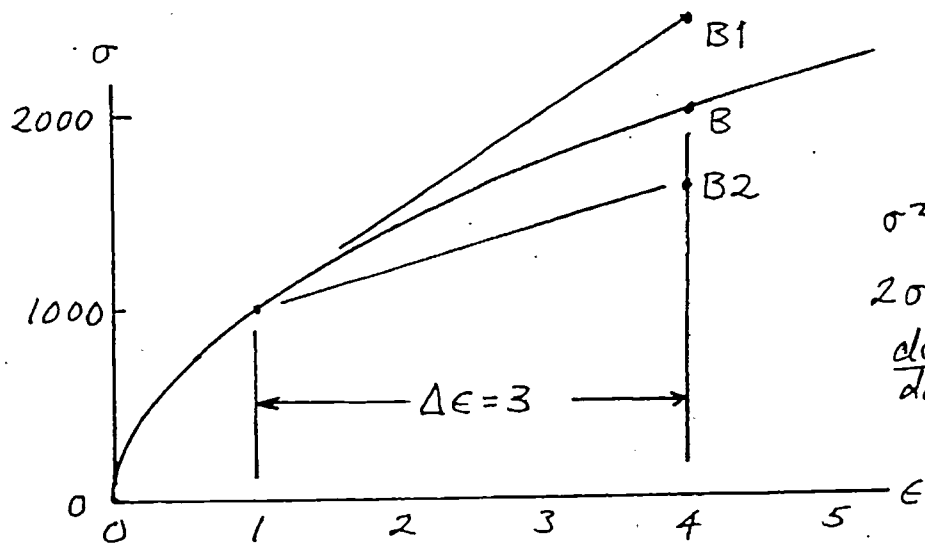
$$(F_i)_{\text{yield}} = (F_i)_{\text{previous}} + r_i (\Delta F_i)$$

Scale the solution update by multiplying it by the smallest r_i , then update.

Repeat to fail one more bar. Structure collapses when $[K]$ becomes singular.

(If the truss is statically determinate initially, it will collapse when the first bar yields or buckles.)

17.3-6



$$\Delta\sigma_1 = \frac{10^6}{2\sigma} \Delta\epsilon = \frac{10^6}{2(1000)} 3 = 1500$$

$$\sigma_{B1} = 1000 + 1500 = 2500$$

$$\Delta\sigma_2 = \frac{10^6}{2(2500)} 3 = 600$$

$$\text{Now Eq. 17.3-7: } \sigma_B \approx 1000 + \frac{1}{2}(1500 + 600) = 2050 \quad (2.5\% \text{ high})$$

Exact σ_B is 2000

17.5-1

(a) Gauss rule does not detect start of yielding: no sampling points at ends of interval (surfaces), where yield begins

(b) Let $d=2$, width = 1. Exact M is

$$M = 2 \left[\left(\sigma_a \frac{1}{2} \right) \frac{3}{4} + \left(\frac{\sigma_a}{2} \frac{1}{2} \right) \left(\frac{2}{3} \frac{1}{2} \right) \right] = 0.91667 \sigma_a$$

Two Gauss pts.: $M_c = \int_{-1}^1 \sigma(1)y dy = 2 \sigma_{GP} y_{GP}$

$$M_c = 2 \sigma_a \frac{\sqrt{3}}{3} = 1.1547 \sigma_a \quad (26.0\% \text{ high})$$

Three Gauss pts.:

$$M_c = 2 \left(\frac{5}{9} \sigma_a \sqrt{0.6} \right) = 0.8607 \sigma_a \quad (6.1\% \text{ low})$$

(c) $\int_a^b f(y) dy = \frac{b-a}{n} \left(\frac{1}{2} f_0 + f_1 + f_2 + \dots + f_{n-1} + \frac{1}{2} f_n \right)$

$$M_{c3} = \frac{2}{2} \left(\frac{1}{2} \sigma_a + 0 + \frac{1}{2} \sigma_a \right) = \sigma_a \quad (9.1\% \text{ high})$$

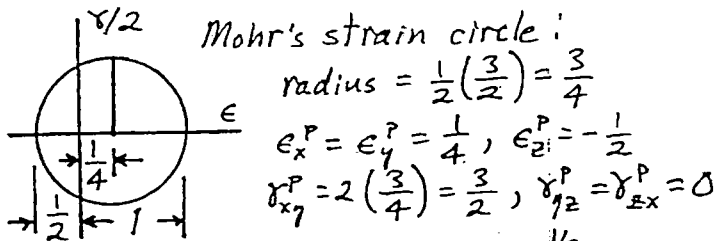
$$M_{c5} = \frac{2}{4} \left(\frac{1}{2} \sigma_a + \sigma_a \frac{1}{2} + 0 + \sigma_a \frac{1}{2} + \frac{1}{2} \sigma_a \right) = \sigma_a$$

$$M_{c7} = \frac{2}{6} \left(\frac{1}{2} \sigma_a + \sigma_a \frac{2}{3} + \frac{2\sigma_a}{3} \frac{1}{3} + 0 + \frac{2\sigma_a}{3} \frac{1}{3} + \sigma_a \frac{2}{3} + \frac{1}{2} \sigma_a \right) = 0.9259 \sigma_a \quad (1.0\% \text{ hi})$$

$$M_{c9} = \frac{2}{8} \left(\frac{1}{2} \sigma_a + \sigma_a \frac{3}{4} + \sigma_a \frac{1}{2} + \frac{\sigma_a}{2} \frac{1}{4} + 0 + \frac{\sigma_a}{2} \frac{1}{4} + \sigma_a \frac{1}{2} + \sigma_a \frac{3}{4} + \frac{1}{2} \sigma_a \right) = 0.9375 \sigma_a \quad (2.3\% \text{ hi})$$

17.5-2

In what follows, ϵ and γ represent strain increments.



$$\epsilon_{ef}^P = \frac{\sqrt{2}}{3} \left[0^2 + \left(\frac{3}{4} \right)^2 + \left(\frac{3}{4} \right)^2 + \frac{3}{2} \left(\frac{3}{2} \right)^2 \right]^{1/2}$$

$$\epsilon_{ef}^P = \frac{\sqrt{2}}{3} \left[\frac{18}{16} + \frac{27}{8} \right]^{1/2} = \frac{\sqrt{2}}{3} \left[\frac{36}{8} \right]^{1/2} = \frac{\sqrt{2}}{3} \frac{6}{2\sqrt{2}} = 1$$

- 1. Apply a load; calculate elastic solution & yield function F_i at each sampling point i (all $F_i < 0$ if no yielding). Calculate factor that will place most highly stressed sampling point at yield. Multiply elastic solution by this factor.
- 2. Determine $[E_{ep}]$ for sampling points that have yielded or will yield in next step. Form $[K_t] = \sum [k]$, using $[E]$ or $[E_{ep}]$ as appropriate for the various sampling points.
- 3. Apply a load increment; solve for increments of displacement and stress. Write Eq. 17.4-13 in form $F = S - \sigma_Y$. For each sampling point not yet yielded, calculate factor r that will initiate yield:

$$\sigma_Y = S_{\text{previous}} + r \Delta S_{\text{current}}$$
 Scale solution increment by smallest r and update the solution using the scaled increment. Go to step 2.

17.5-4

Similar to Problem 17.5-3. Calculate σ_{max} at each sampling point. In Step 1, find sampling point with largest σ_{max} ; compute $r = \sigma_t / \sigma_{max}$; scale solution by r . In Step 2, calculate $[E_{ep}]$ as follows.

$$\text{Let } [E'] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & \beta G \end{bmatrix} \begin{matrix} \sigma_{max} \\ \sigma_{min} \\ \sigma \end{matrix} \quad \begin{matrix} \text{(referred to} \\ \text{prin. stress} \\ \text{directions)} \end{matrix}$$

where $0 < \beta < 1$ to retain partial shear stiffness. Also, set $E = 0$ in σ_{max} and σ_{min} exceed σ_t . Transform to global directions, $[E_{ep}] = [I]^T [E'] [I]$.

In Step 3, use the equation

$$\sigma_t = (\sigma_{max})_{previous} + r(\Delta\sigma_{max})_{current}$$

Collapse indicated by very large $\{\Delta D\}$ or by unsolvable equations.

17.9-1

Green strain: $\epsilon_x = \frac{u_z}{L} + \frac{1}{2} \left(\frac{u_z^2}{L^2} + \frac{v_z^2}{L^2} \right)$

Let $r = \frac{u_z}{L}$, which is the engineering definition of strain.

For Green strain to be 105% of $\frac{u_z}{L}$, $0.05r = \frac{1}{2} \left(r^2 + \frac{v_z^2}{L^2} \right)$ (A)

(a) $v_z = 0$; (A) gives $r = 0.10$

(b) $v_z = u_z$; (A) becomes $0.05r = r^2$, so $r = 0.05$

(c) $v_z = 100u_z$; (A) becomes $0.05r = \frac{1}{2}(101r^2)$, so $r = \frac{0.1}{101} \approx 0.001$

If instead we set Green strain to be 95% of $\frac{u_z}{L}$, we obtain the same magnitudes of r but with a negative sign on each.

17.9-2

$$\epsilon_x = u_{,x} + \frac{1}{2}(u_{,x}^2 + v_{,x}^2)$$

$$\epsilon_y = v_{,y} + \frac{1}{2}(u_{,y}^2 + v_{,y}^2)$$

$$\gamma_{xy} = u_{,y} + v_{,x} + (u_{,x}u_{,y} + v_{,x}v_{,y})$$

$$u = a_1 + (\cos \theta - 1)x - (\sin \theta)y$$

$$v = a_2 + (\sin \theta)x + (\cos \theta - 1)y$$

$$\epsilon_x = \cos \theta - 1 + \frac{1}{2}(\cos^2 \theta - 2 \cos \theta + 1 + \sin^2 \theta)$$

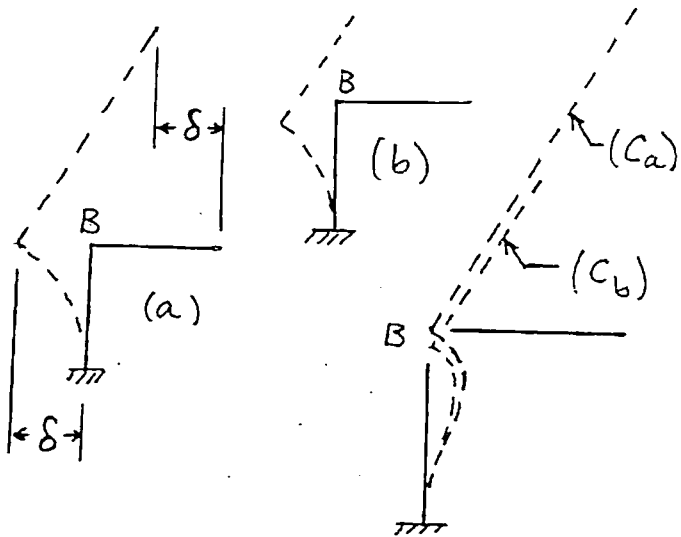
$$= \cos \theta - 1 + (1 - \cos \theta) = 0$$

$$\epsilon_y = \cos \theta - 1 + \frac{1}{2}(\sin^2 \theta + \cos^2 \theta - 2 \cos \theta + 1)$$

$$= \cos \theta - 1 + (1 - \cos \theta) = 0$$

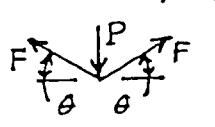
$$\gamma_{xy} = -\sin \theta + \sin \theta + (\cos \theta - 1)(-\sin \theta) + (\sin \theta)(\cos \theta - 1) = 0$$

17.9-3



17.9-4

$$P = 2F \sin \theta, F = k_0 (\sqrt{L^2 + D^2} - L)$$



$$\sin \theta = \frac{D}{\sqrt{L^2 + D^2}}$$

Hence $P = 2k_0 D \frac{\sqrt{L^2 + D^2} - L}{\sqrt{L^2 + D^2}}$ or

$$P = 2k_0 D \frac{\sqrt{1 + D^2/L^2} - 1}{\sqrt{1 + D^2/L^2}} \quad (A)$$

For small D , $\frac{1}{1 + \frac{D^2}{L^2}} - 1$

$$P = 2k_0 D \frac{\frac{D^2}{2L^2}}{1 + \frac{D^2}{2L^2}} = 2k_0 D \frac{D^2}{2L^2} = k_0 \frac{D^3}{L^2} \quad (B)$$

$$k_{\text{secant}} = P/D = \frac{k_0 D^2}{L^2}$$

(b) From Eq. (B), $\frac{dP}{dD} = 3k_0 \frac{D^2}{L^2} = k_t$

(c) From Eq. (A), $\frac{P}{D} = 2k_0 \frac{\sqrt{1 + D^2/L^2} - 1}{\sqrt{1 + D^2/L^2}}$

(d) At $D = \frac{1}{2}$, $k_t = \frac{3(800)(\frac{1}{2})^2}{10^2} = 6$

Apply $\Delta P = 7$ in $k_t \Delta D = \Delta P$

$6 \Delta D = 7$, $\Delta D = 1.167$, $D = \frac{1}{2} + \Delta D$
 $D = 1.667$

Equilibrium iterations:

$$k_t = \frac{3(800)(1.667)^2}{10^2} = 66.67$$

$$\Delta D = -0.436, D = 1.231$$

$$k_t = \frac{3(800)(1.231)^2}{10^2} = 36.37$$

$$\Delta D = -0.190, D = 1.041$$

$$k_t = \frac{3(800)(1.041)^2}{10^2} = 25.99$$

$$\Delta D = -0.039, D = 1.002$$

$\Delta D = -0.039$, $D = 1.002$

From eq. (B),

$$D = \frac{P L^2}{k_0} = \frac{8(10)^2}{800} = 1, D = 1$$

17.9-5

Let α be the rigid-body rotation angle.

$$\beta_0 = \gamma_0 = D_1 = D_2 = 0$$

$$x_0 = L = L_0$$

$$D_3 = D_6 = \alpha$$

$$D_5 = L_0 \sin \alpha$$

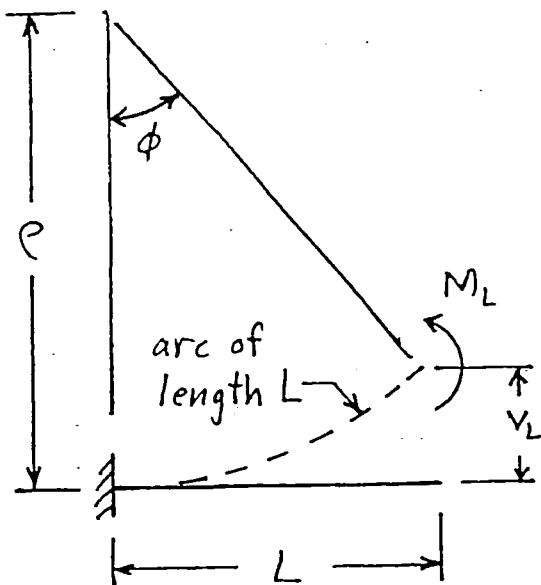
$$D_4 = -L_0(1 - \cos \alpha)$$

$$u_2 = \frac{1}{2L_0} [2L_0 - L_0(1 - \cos \alpha)] [-L_0(1 - \cos \alpha)] + \frac{1}{2L_0} [L_0 \sin \alpha] [L_0 \sin \alpha]$$

$$u_2 = \frac{L_0}{2} [-2(1 - \cos \alpha) + (1 - 2\cos \alpha + \cos^2 \alpha) + \sin^2 \alpha]$$

$$u_2 = \frac{L_0}{2} [-1 + \cos^2 \alpha + \sin^2 \alpha] = 0$$

17.9-6



$$\left. \begin{aligned} \frac{1}{\rho} &= \frac{M}{EI} \\ \phi &= \frac{L}{\rho} = \frac{ML}{EI} \end{aligned} \right\} v_L = \rho(1 - \cos \phi) = \frac{EI}{M} \left(1 - \cos \frac{ML}{EI} \right)$$

$$\cos \phi = 1 - \frac{\phi^2}{2} + \frac{\phi^4}{24} - \dots$$

$$\text{For small } \phi, \quad 1 - \cos \phi \approx \frac{\phi^2}{2}$$

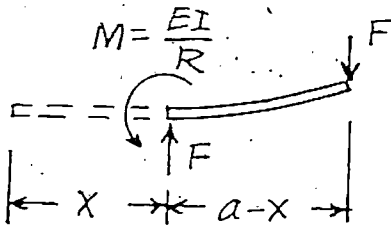
$$\text{Thus } 1 - \cos \frac{ML}{EI} \text{ becomes } \frac{1}{2} \left(\frac{ML}{EI} \right)^2, \text{ and}$$

$$v_L \approx \frac{1}{M} \frac{1}{2} \left(\frac{ML}{EI} \right)^2 = \frac{ML^2}{2EI}$$

17.9-7

Becomes straight when change of curvature is $\frac{1}{R}$. But $\frac{1}{R} = \frac{M}{EI}$,
so at left end, where $M = Fa$, $\frac{1}{R} = \frac{Fa}{EI}$ and $F = \frac{EI}{Ra}$

For $F > \frac{EI}{Ra}$;

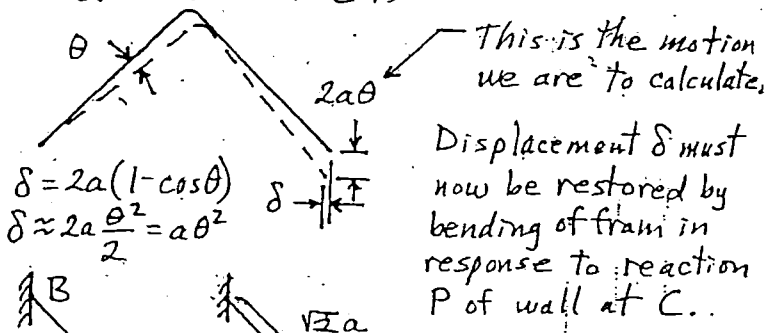


$$M = F(a-x)$$

$$x = a - \frac{M}{F} = a - \frac{EI}{FR}$$

17.9-8

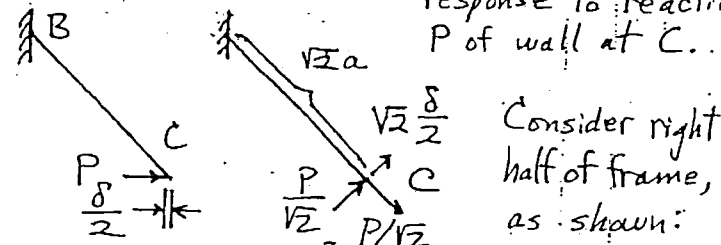
Let θ = rigid body rotation about A, with no force at C. Motion of C is



$$\delta = 2a(1 - \cos\theta)$$

$$\delta \approx 2a \frac{\theta^2}{2} = a\theta^2$$

Displacement δ must now be restored by bending of frame in response to reaction P of wall at C.



$$\frac{\sqrt{2}\delta}{2} = \frac{(P/\sqrt{2})(\sqrt{2}a)^3}{3EI}, \quad \delta = a\theta^2 = \frac{2\sqrt{2}Pa^3}{3EI}$$

$$P = \frac{3EI\theta^2}{2\sqrt{2}a^2}. \text{ Moments about A: } Fa = (2a\theta)P,$$

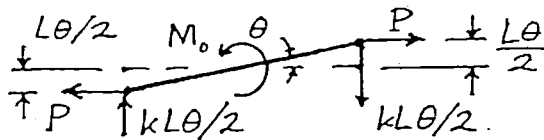
$$\text{hence } F = 2\theta P = \frac{3EI\theta^3}{\sqrt{2}a^2}, \quad \theta = \left(\frac{\sqrt{2}Fa^2}{3EI} \right)^{1/3}$$

$$2a\theta = \left(8a^3 \frac{\sqrt{2}Fa^2}{3EI} \right)^{1/3} = \left(\frac{8\sqrt{2}Fa^5}{3EI} \right)^{1/3}$$

18.1-1

No. Force in rod (tension) = force in column (comp.). During a virtual lateral displacement, these forces do work of equal magnitude but opposite sign; i.e. the net work is zero. No membrane energy is lost, so no bending energy is to be gained, $\therefore$ no buckling.

18.1-2



$$(a) \Pi_P = 2 \left[\frac{1}{2} k \left(\frac{L\theta}{2} \right)^2 \right] + P \left(\frac{1}{2} L \theta^2 \right) - M_0 \theta$$

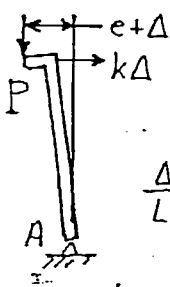
$$\frac{d\Pi_P}{d\theta} = 0 = \frac{kL^2}{2} \theta + PL\theta - M_0$$

$$\theta = \frac{M_0}{\frac{kL^2}{2} + PL}$$

(b) $\theta \rightarrow \infty$ if $\frac{kL^2}{2} + PL = 0$, so $P_{cr} = -\frac{kL}{2}$
or, discard M_0 from Π_P , solve for P .

18.1-3

$$\Sigma M_A = 0 = P(c + \Delta) - k\Delta(L)$$

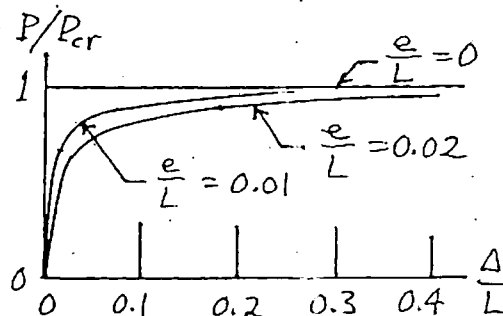


$$\Delta(kL - P) = 0, \Delta = \frac{Pe}{kL - P}$$

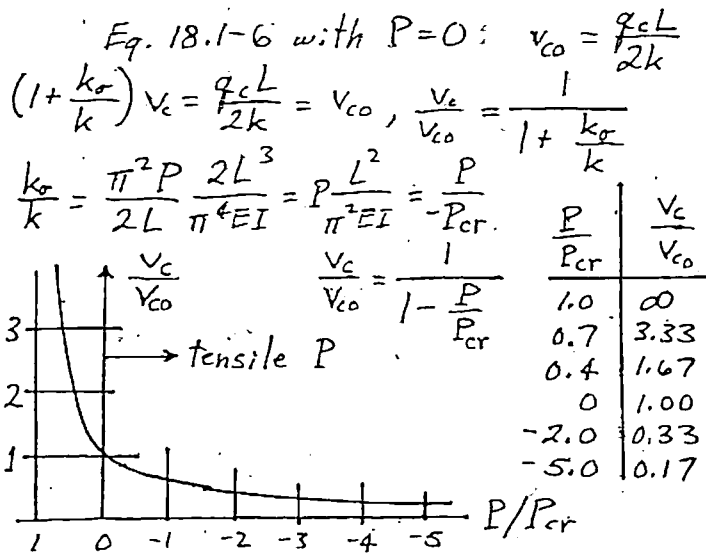
$$\Delta \rightarrow \infty \text{ as } kL - P \rightarrow 0; P_{cr} = kL$$

$$\frac{\Delta}{L} = \frac{\frac{P}{kL} \frac{e}{L}}{1 - \frac{P}{kL}} = \frac{\frac{P}{P_{cr}} \frac{e}{L}}{1 - \frac{P}{P_{cr}}} = f_P \frac{e}{L}$$

P/P_{cr}	f_P
0.4	0.667
0.7	2.33
0.9	9.00
0.95	19.0



18.1-4



18.2-1

Given eqs. to start with are

$$\epsilon_x = u_{,x} + \frac{1}{2} v_{,x}^2 - y v_{,xx}$$

$$U = \int \frac{1}{2} E \epsilon_x^2 dV$$

Substitute 1st eq. into 2nd eq.; also $dV = dA dx$

$$U = \frac{E}{2} \int \left(u_{,x}^2 + \frac{1}{4} v_{,x}^4 + y^2 v_{,xx}^2 + u_{,x} v_{,x}^2 - y v_{,x}^2 v_{,xx} - 2 y u_{,x} v_{,xx} \right) dA dx$$

With $y=0$ at centroidal axis of A , terms linear in y integrate to zero over A . Also $\int E u_{,x} dA = P$ and $\int y^2 dA = I$. Thus

$$U = \frac{1}{2} \int u_{,x}^2 E A dx + \frac{1}{4} \int v_{,x}^4 E A dx + \frac{1}{2} \int v_{,xx}^2 E I dx + \frac{1}{2} \int v_{,x}^2 P dx$$

Discard the second integral as negligible in comparison with others. Also interpolate displacements in terms of nodal d.o.f. $\underline{d}$.

$$u = \underline{N}_u \underline{d}_u \quad v = \underline{N}_v \underline{d}_v$$

$$U = \frac{1}{2} \underline{d}_u^T \int \underline{N}_{u,x}^T \underline{N}_{u,x} E A dx \underline{d}_u + \frac{1}{2} \underline{d}_v^T \int \underline{N}_{v,xx}^T \underline{N}_{v,xx} E I dx \underline{d}_v + \frac{1}{2} \underline{d}_v^T \int \underline{N}_{v,x}^T \underline{N}_{v,x} P dx \underline{d}_v$$

or

$$U = \frac{1}{2} \underline{d}_u^T \underline{k}_{bar} \underline{d}_u + \frac{1}{2} \underline{d}_v^T \underline{k}_{beam} \underline{d}_v + \frac{1}{2} \underline{d}_v^T \underline{k}_r \underline{d}_v$$

18.2-2

In the formula for $[k]$, Eq. 3.3-14, EI becomes a function of x . The formula for $[k_\sigma]$, Eq. 18.2-3, says nothing about geometry of the cross section. Therefore $[k_\sigma]$ is not affected by taper if $[N]$ is not changed.

18.2-3

$$\frac{d}{dx} = \frac{1}{L} \frac{d}{d\xi}, \quad \frac{d}{dx} \left[\frac{L}{2} (\xi - \xi^2) \right] = \frac{1}{2} - \xi = \frac{1}{2} - \frac{x}{L}$$

$$v_x = \underbrace{\left[-\frac{1}{L}, \frac{1}{2} - \frac{x}{L}, \frac{1}{L}, -\frac{1}{2} + \frac{x}{L} \right]}_{[G]} [v_1 \theta_1 v_2 \theta_2]^T$$

$$\int_0^L \frac{1}{L^2} dx = \frac{1}{L}, \quad \int_0^L \frac{1}{L} \left(\frac{1}{2} - \frac{x}{L} \right) dx = 0, \quad \int_0^L \left(\frac{1}{2} - \frac{x}{L} \right)^2 dx = \frac{L}{12}$$

Hence

$$[k_\sigma] = \int_0^L [G]^T P [G] dx = \frac{P}{12L} \begin{bmatrix} 12 & 0 & -12 & 0 \\ 0 & L^2 & 0 & -L^2 \\ -12 & 0 & 12 & 0 \\ 0 & -L^2 & 0 & L^2 \end{bmatrix}$$

18.2-4

Impose rigid body rotation (no curvature). That is, impose rotation d.o.f. $\theta_1 = \theta_2 = \frac{v_2 - v_1}{L}$.

Thus, for $[k_\sigma]$, use $\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = [T] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$, where

$$[T] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1/L & 0 & 1/L \\ 0 & 0 & 0 & 1 \\ 0 & -1/L & 0 & 1/L \end{bmatrix}$$

$$[T]^T ([k_\sigma] [T]) = \frac{P}{30L} = \frac{P}{L} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

18,2-5

As rigid bar rotates to angle β , where $\beta = v_2/c$, it exerts vertical force $\beta P = v_2 P/c$ on node 2.

$$([k] + [k_\theta])\{D\} = \{R\}$$

$$\left(\frac{EI}{L^3} \begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix} - \frac{P}{30L} \begin{bmatrix} 36 & -3L \\ -3L & 4L^2 \end{bmatrix} \right) \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} Q + P \frac{v_2}{c} \\ 0 \end{Bmatrix}$$

$$\left(\frac{EI}{L^3} \begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix} - \frac{P}{30L} \begin{bmatrix} 36 + 30 \frac{L}{c} & -3L \\ -3L & 4L^2 \end{bmatrix} \right) \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} Q \\ 0 \end{Bmatrix}$$

18,2-6

For small rotation v_2/L of bar 1-2, bar 1-2 carries force P and exerts downward force $P v_2/L$ on node 2. Sum of vertical forces on node 2 yields

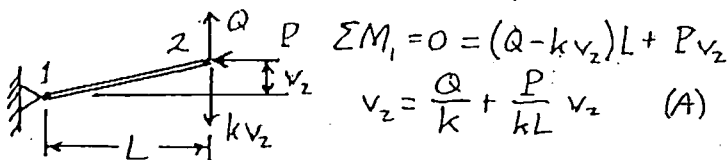
$$Q - kv_2 - \frac{P}{L} v_2 = 0, \quad v_2 = \frac{Q}{k + \frac{P}{L}}$$

(a) $P=0, v_2 = Q/k$

(b) $P=0.96kL, v_2 = 0.510 Q/k$

(c) $P=-0.96kL, v_2 = 25.0 Q/k$

18,2-7



$$\sum M_1 = 0 = (Q - kv_2)L + Pv_2$$

$$v_2 = \frac{Q}{k} + \frac{P}{kL} v_2 \quad (A)$$

Set $P=0.96kL$ & write (A) in iterative form

$$(v_2)_i = \frac{Q}{k} + 0.96(v_2)_{i-1}$$

i	1	2	3	---	∞
$(v_2)_i$	Q/k	$1.96Q/k$	$2.8816Q/k$	---	$25.0Q/k$

18.2-8

(a) θ_2 is the only d.o.f.; $M_2 = P_e$ the only load.

$$\left(\frac{EI}{L^3} 4L^2 - \frac{P}{30L} 4L^2 \right) \theta_2 = P_e$$

$$\theta_2 = \left(\frac{4EI}{L} - \frac{PL}{7.5} \right)^{-1} P_e$$

(b) Set $\left(\frac{4EI}{L} - \frac{PL}{7.5} \right) = 0$, get $P_{cr} = \frac{30EI}{L^2}$

This is $\approx 50\%$ high (exact $P_{cr} = \frac{20.2EI}{L^2}$)

18.2-9

Note: $P_{cr} = -\frac{\pi^2 EI}{4L^2} = -30.157 \cdot N$

(a) The one element gives the exact result,

$$w_2 = \frac{0.1 L^3}{3EI} = 0.008182 \text{ m} \quad (\text{for } P=0)$$

(b) Standard beam $[k]$, $[k_r]$ from Eq. 18.2-6:

$$\left(\frac{110}{27} \begin{bmatrix} 12 & -6(3) \\ -6(3) & 4(9) \end{bmatrix} + \frac{-30}{30(3)} \begin{bmatrix} 36 & -3(3) \\ -3(3) & 4(9) \end{bmatrix} \right) \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0.1 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 36.889 & -70.333 \\ -70.333 & 134.667 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0.1 \\ 0 \end{Bmatrix} \quad \text{Solving,}$$

$$v_2 = \frac{0.1}{36.889 - 36.734} = 0.645 \text{ m}$$

(c) In the second matrix of part (b), change

-30 to +30. Thus $\begin{bmatrix} 60.889 & -76.333 \\ -76.333 & 158.667 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0.1 \\ 0 \end{Bmatrix}$

$$v_2 = \frac{0.1}{60.889 - 36.723} = 0.00414 \text{ m}$$

8.2-10

(a) — As before; $v_2 = 0.008182 \text{ m}$

$$(b) \left(\frac{110}{27} \begin{bmatrix} 12 & -18 \\ -18 & 36 \end{bmatrix} + \frac{-36}{36} \begin{bmatrix} 12 & 0 \\ 0 & 9 \end{bmatrix} \right) \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0.1 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 4200 & -7920 \\ -7920 & 15,030 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 10.8 \\ 0 \end{Bmatrix} \text{ Solving,}$$

$$v_2 = \frac{1}{(63.1260 - 62.7264)(10)^6} \begin{bmatrix} 15,030 & 7920 \\ 7920 & 4200 \end{bmatrix} \begin{Bmatrix} 10.8 \\ 0 \end{Bmatrix}$$

$$v_2 = 0.406 \text{ m}$$

(c) In the second matrix of part (b), change

$$-30 \text{ to } +30. \text{ Thus } \begin{bmatrix} 6360 & -7920 \\ -7920 & 16,650 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 10.8 \\ 0 \end{Bmatrix}$$

$$v_2 = \frac{1}{(105,894 - 62.7264)(10)^6} \begin{bmatrix} 16,650 & 7920 \\ 7920 & 6360 \end{bmatrix} \begin{Bmatrix} 10.8 \\ 0 \end{Bmatrix}$$

$$v_2 = 0.00417 \text{ m}$$

18.2-11

From [B] in Eq. 3.3-13 at $x=0$,

$$M_0 = EI(v_{xx})_{x=0} = EI \begin{bmatrix} -\frac{6}{L^2} & -\frac{4}{L} & \frac{6}{L^2} & -\frac{2}{L} \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$v_1 = \theta_1 = 0$$

$$v_2 = 0.645 \text{ m}, \theta_2 = \frac{70.333}{134.667} \quad v_2 = 0.337$$

$$M_0 = 110 \begin{bmatrix} \frac{6}{9} & -\frac{2}{3} \end{bmatrix} \begin{Bmatrix} 0.645 \\ 0.337 \end{Bmatrix} = 22.6 \text{ N}\cdot\text{m}$$

By force * displacement,

$$M_0 = 0.1(3) + 30(0.645) = 19.7 \text{ N}\cdot\text{m}$$

18.2-12

We must obtain $[k_\sigma]\{d\} = \{0\}$ if $\{d\}$ represents rigid-body translation.

Hence, if $[k_\sigma]$ is diagonal & its

non-zero terms operate only on

translational d.o.f., $[k_\sigma]$ must be null.

18.4-1

$\{\underline{\delta}\} = [\underline{G}]\{\underline{d}\}$, where $\{\underline{\delta}\}$ is given by Eq. 18.4-3, and $\{\underline{d}\} = [u, u_2 \dots u_8, v, \dots w_7, w_8]^T$.

Also $u = \sum_1^8 N_i u_i$ $v = \sum_1^8 N_i v_i$ $w = \sum_1^8 N_i w_i$

$\begin{Bmatrix} u_{,x} \\ u_{,y} \\ u_{,z} \end{Bmatrix} = [\underline{G}_r] \begin{Bmatrix} u_1 \\ \vdots \\ u_8 \end{Bmatrix}$ and similarly for derivatives of v and w , where

$$[\underline{G}_r] = \begin{bmatrix} N_{1,x} & N_{2,x} & \dots & N_{8,x} \\ N_{1,y} & N_{2,y} & \dots & N_{8,y} \\ N_{1,z} & N_{2,z} & \dots & N_{8,z} \end{bmatrix}, [\underline{G}] = \begin{bmatrix} \underline{G}_r & 0 & 0 \\ 0 & \underline{G}_r & 0 \\ 0 & 0 & \underline{G}_r \end{bmatrix}$$

$$N_{i,x} = \Gamma_{11} N_{i,\xi} + \Gamma_{12} N_{i,\eta} + \Gamma_{13} N_{i,\zeta}$$

$$N_{i,y} = \Gamma_{21} N_{i,\xi} + \Gamma_{22} N_{i,\eta} + \Gamma_{23} N_{i,\zeta}$$

$$N_{i,z} = \Gamma_{31} N_{i,\xi} + \Gamma_{32} N_{i,\eta} + \Gamma_{33} N_{i,\zeta}$$

18.4-2

(a) Discard terms in $[\underline{S}]$ with z subscripts. Also, $w=0$; only u & v have derivatives w.r.t. x & y .

$$\{\underline{\delta}\} = \begin{Bmatrix} u_{,x} \\ u_{,y} \\ v_{,x} \\ v_{,y} \end{Bmatrix} = [\underline{G}] \{\underline{d}\} = \begin{bmatrix} \underline{B} & \underline{0} \\ \underline{0} & \underline{B} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{Bmatrix}, \text{ where,}$$

from Eq. 7.2-6,

$$[\underline{B}] = \frac{1}{2A} \begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix}$$

$$[\underline{S}] = \begin{bmatrix} \tau_{x0} & \tau_{xy0} \\ \tau_{xy0} & \sigma_{y0} \end{bmatrix}$$

$$[\underline{k}_\sigma] = \iint \begin{bmatrix} \underline{B}^T & \underline{0} \\ \underline{0} & \underline{B}^T \end{bmatrix} \begin{bmatrix} \underline{S} & \underline{0} \\ \underline{0} & \underline{S} \end{bmatrix} \begin{bmatrix} \underline{B} & \underline{0} \\ \underline{0} & \underline{B} \end{bmatrix} t \, dx \, dy$$

$$(b) [\underline{k}_\sigma] = \iint \begin{bmatrix} \underline{B}^T & \underline{0} \\ \underline{0} & \underline{B}^T \end{bmatrix} \begin{bmatrix} \underline{S} & \underline{0} \\ \underline{0} & \underline{S} \end{bmatrix} \begin{bmatrix} \underline{B} & \underline{0} \\ \underline{0} & \underline{B} \end{bmatrix} t \, dx \, dy, \{\underline{d}\} = \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}$$

18.5-1

We will solve for both roots. Expansion of the determinant by cofactors shows at once that only the southeast 2 by 2 submatrices need be used. Let $s = \frac{\lambda_{cr} L^3}{30L^2 EI} = \frac{\lambda_{cr} L}{60EI}$.

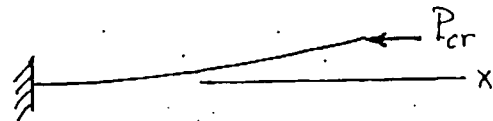
$$\begin{vmatrix} 6-36s & -3L+3Ls \\ -3L+3Ls & 2L^2-4sL^2 \end{vmatrix} = 45s^2 - 26s + 1 = 0$$

$$s = \frac{26 \pm \sqrt{496}}{90} \Rightarrow \begin{cases} s_1 = 0.04143, \lambda_1 = 2.486 \frac{EI}{L^2} \\ s_2 = 0.53635, \lambda_2 = 32.18 \frac{EI}{L^2} \end{cases}$$

Mode, with $\theta_2 = 1$ and $s = s_1$:

$$(6-36s_1)v_2 + (-3L+3Ls_1)(1) = 0$$

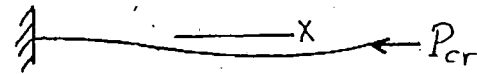
$$v_2 = \frac{1-s_1}{2-12s_1} L = 0.6379L$$



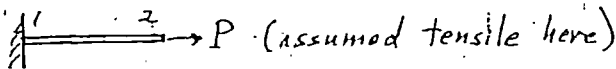
Mode, with $\theta_2 = 1$ and $s = s_2$:

$$(6-36s_2)v_2 + (-3L+3Ls_2)(1) = 0$$

$$v_2 = \frac{1-s_2}{2-12s_2} L = -0.1045L$$



18.5-2



$$\left(\frac{EI}{L^3} \begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix} + \frac{P}{12L} \begin{bmatrix} 12 & 0 \\ 0 & L^2 \end{bmatrix} \right) \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Let } s = \frac{PL^2}{24EI}, \text{ then } \begin{vmatrix} 6+12s & -3L \\ -3L & 2L^2+sL^2 \end{vmatrix} = 0$$

$$12s^2 + 30s + 3 = 0, s = -0.10436, P_{cr} = -2.505 \frac{EI}{L^2}$$

18.5-3

(a) Use $[k_r]$ from Eq. 18.2-4.

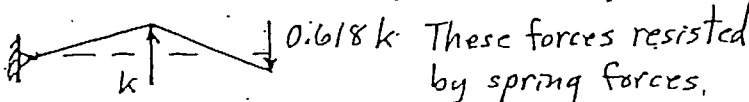
$$\left(\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} - \frac{P_{cr}}{a} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \right) \begin{Bmatrix} v_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \text{ Let } s = \frac{P_{cr}}{a}$$

$$\begin{vmatrix} k-2s & s \\ s & k-s \end{vmatrix} = s^2 - 3ks + k^2 = 0, s_{cr} = \frac{3-\sqrt{5}}{2} k$$

$$P_{cr} = 0.382 ka$$

$$(b) k[v_2 - 0.382(2v_2 - v_3)] = 0; \text{ set } v_2 = 1, \text{ then } v_3 = -0.618$$

$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{Bmatrix} v_2 \\ v_3 \end{Bmatrix} = 0.382 \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ -0.618 \end{Bmatrix} = k \begin{Bmatrix} 1 \\ -0.618 \end{Bmatrix}$$



18.5-4

(a)

$$\left[\frac{EI}{a^3} \begin{bmatrix} 24 & 0 & 6a \\ 0 & 8a^2 & 2a^2 \\ 6a & 2a^2 & 4a^2 \end{bmatrix} - \frac{P}{30a} \begin{bmatrix} 72 & 0 & 3a \\ 0 & 8a^2 & -a^2 \\ 3a & -a^2 & 4a^2 \end{bmatrix} \right] \begin{Bmatrix} v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \\ 0 \end{Bmatrix}$$

$$[T] = \begin{bmatrix} 1 & & \\ -\frac{1}{28a^4} \begin{bmatrix} 4a^2 & -2a^2 \\ -2a^2 & 8a^2 \end{bmatrix} & \begin{Bmatrix} 0 \\ 6a \end{Bmatrix} & \end{bmatrix} = \begin{Bmatrix} 1 \\ 3/7a \\ -12/7a \end{Bmatrix}$$

Condensed arrays $[T]^T[\cdot][T]$ are

$$\frac{EI}{a^3} \begin{bmatrix} 1 & \frac{3}{7a} & -\frac{12}{7a} \end{bmatrix} \begin{Bmatrix} 13.7143 \\ 0 \\ 0 \end{Bmatrix} = \frac{13.7143 EI}{a^3}$$

$$\frac{P}{30a} \begin{bmatrix} 1 & \frac{3}{7a} & -\frac{12}{7a} \end{bmatrix} \begin{Bmatrix} 66.8571 \\ 5.1429a \\ -4.2857a \end{Bmatrix} = \frac{2.5469 P}{a}$$

$$\left(\frac{13.7143 EI}{a^3} - \frac{2.5469 P}{a} \right) v_2 = F \quad (A)$$

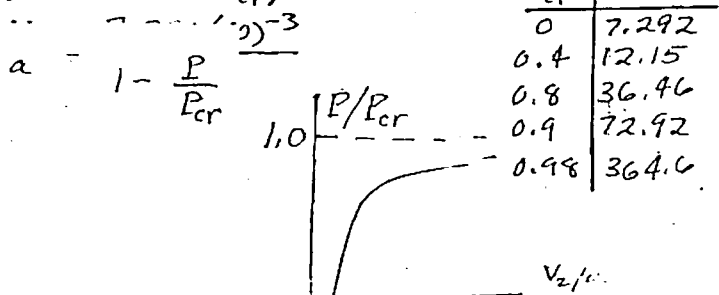
$$(b) F=0, P_{cr} = \frac{13.7143}{2.5469} \frac{EI}{a^2} = 5.385 \frac{EI}{a^2} \quad (B)$$

$$\text{exact } P_{cr} = 20.2 \frac{EI}{L^2} = 5.05 \frac{EI}{a^2} \quad \uparrow 6.6\% \text{ high}$$

(c) Divide (A) by P_{cr} from (B).

$$\left(\frac{13.7143 EI/a^2}{5.385 EI/a^2} - \frac{2.5469 P}{P_{cr}} \right) \frac{v_2}{a} = \frac{0.1 EI/a^2}{5.385 EI/a^2}$$

$$\left(2.547 - 2.547 \frac{P}{P_{cr}} \right) \frac{v_2}{a} = 0.01857 \quad \frac{P}{P_{cr}} \quad \left| \quad \frac{v_2}{a} \times 10^3 \right.$$



18.5-5

(a)

$$\left(\frac{EI}{L^3} \begin{bmatrix} 4L^2 & 2L^2 \\ 2L^2 & 4L^2 \end{bmatrix} + \frac{P}{30L} \begin{bmatrix} 4L^2 & -L^2 \\ -L^2 & 4L^2 \end{bmatrix} \right) \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Let } s = PL^2/60EI$$

$$\begin{vmatrix} 2+4s & 1-s \\ 1-s & 2+4s \end{vmatrix} = 15s^2 + 18s + 3 = 0, s = -0.2$$

$$P_{cr} = -\frac{12EI}{L^2}$$

(b) Let $a = L/2$

$$\left(\frac{EI}{a^3} \begin{bmatrix} 4a^2 & -6a \\ -6a & 12 \end{bmatrix} + \frac{P}{30a} \begin{bmatrix} 4a^2 & -3a \\ -3a & 36 \end{bmatrix} \right) \begin{Bmatrix} \theta_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Let } s = Pa^2/60EI$$

$$\begin{vmatrix} 2a^2 + 4sa^2 & -3a - 3as \\ -3a - 3as & 6 + 36s \end{vmatrix} = (135s^2 + 78s + 3)a^2 = 0$$

$$s = -0.0414327, P_{cr} = -2.4859 \frac{EI}{a^2} = -9.9436 \frac{EI}{L^2}$$

(c)

$$\left(\frac{EI}{L^3} \begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix} + \begin{bmatrix} \frac{2EI}{L^3} & 0 \\ 0 & 0 \end{bmatrix} + \frac{P}{30L} \begin{bmatrix} 36 & -3L \\ -3L & 4L^2 \end{bmatrix} \right) \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Let } s = PL^2/60EI$$

$$\begin{vmatrix} 7+36s & -3L-3Ls \\ -3L-3Ls & 2L^2+4L^2s \end{vmatrix} = (135s^2 + 82s + 5)L^2 = 0$$

$$s = -0.068759, P_{cr} = -4.126 \frac{EI}{L^2}$$

(d)

$$\left(\frac{EI}{L^3} \begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{EI}{L} \end{bmatrix} + \frac{P}{30L} \begin{bmatrix} 36 & -3L \\ -3L & 4L^2 \end{bmatrix} \right) \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Let } s = PL^2/60EI$$

$$\begin{vmatrix} 12+36s & -6L-3Ls \\ -6L-3Ls & 5L^2+4L^2s \end{vmatrix} = (135s^2 + 192s + 24)L^2 = 0$$

$$s = -0.13848, P_{cr} = -4.155 \frac{EI}{L^2}$$

(e) Set $v_2 = 0$ in equations of part (c).

$$\left[\frac{EI}{L^3} (4L^2) + 0 + \frac{P}{30L} (4L^2) \right] \theta_2 = 0, P_{cr} = -30 \frac{EI}{L^2}$$

(f) Set $\theta_2 = 0$ in equations of part (d).

$$\left[\frac{EI}{L^3} (12) + 0 + \frac{P}{30L} (36) \right] v_2 = 0, P_{cr} = -10 \frac{EI}{L^2}$$

18.5-6

(a)

$$\left(\frac{EI}{L^3} \begin{bmatrix} 4L^2 & 2L^2 \\ 2L^2 & 4L^2 \end{bmatrix} + \frac{P}{12L} \begin{bmatrix} L^2 & -L^2 \\ -L^2 & L^2 \end{bmatrix} \right) \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Let } s = PL^2/24EI$$

$$\begin{vmatrix} 2+s & 1-s \\ 1-s & 2+s \end{vmatrix} = 6s+3=0, s=-\frac{1}{2}, P_{cr} = -\frac{12EI}{L^2}$$

(Same as Prob. 18.5-5a because both activate the constant-curvature state, for which $[k_s]$ $[k_{\theta}]$ are the same in these two problems.)

(b) Let $a = L/2$

$$\left(\frac{EI}{a^3} \begin{bmatrix} 4a^2 & -6a \\ -6a & 12 \end{bmatrix} + \frac{P}{12a} \begin{bmatrix} a^2 & 0 \\ 0 & 12 \end{bmatrix} \right) \begin{Bmatrix} \theta_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Let } s = Pa^2/24EI$$

$$\begin{vmatrix} 2a^2+sa^2 & -3a \\ -3a & 6+12s \end{vmatrix} = (12s^2+30s+3)a^2=0$$

$$s = -0.10436, P_{cr} = -2.505 \frac{EI}{a^2} = -10.02 \frac{EI}{L^2}$$

(c)

$$\left(\frac{EI}{L^3} \begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix} + \begin{bmatrix} \frac{2EI}{L^3} & 0 \\ 0 & 0 \end{bmatrix} + \frac{P}{12L} \begin{bmatrix} 12 & 0 \\ 0 & L^2 \end{bmatrix} \right) \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Let } s = PL^2/24EI$$

$$\begin{vmatrix} 7+12s & -3L \\ -3L & 2L^2+L^2s \end{vmatrix} = 12s^2+31s+5=0$$

$$s = -0.17286, P_{cr} = -4.149 \frac{EI}{L^2}$$

(d)

$$\left(\frac{EI}{L^3} \begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{EI}{L} \end{bmatrix} + \frac{P}{12L} \begin{bmatrix} 12 & 0 \\ 0 & L^2 \end{bmatrix} \right) \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Let } s = PL^2/12EI$$

$$\begin{vmatrix} 12+12s & -6L \\ -6L & 5L^2+L^2s \end{vmatrix} = (12s^2+72s+24)L^2=0$$

$$s = -0.354, P_{cr} = -4.251 \frac{EI}{L^2}$$

(e) Set $v_2 = 0$ in equations of part (c).

$$\left[\frac{EI}{L^3} (4L^2) + 0 + \frac{P}{12L} (L^2) \right] \theta_2 = 0, P_{cr} = -48 \frac{EI}{L^2}$$

(f) Set $\theta_2 = 0$ in equations of part (d).

$$\left[\frac{EI}{L^3} (12) + 0 + \frac{P}{12L} (12) \right] v_2 = 0, P_{cr} = -12 \frac{EI}{L^2}$$

18.5-7

(a) A buckling load is not obtained:
 $[k_v]$ that operates on 0 d.o.f. is null.

(b) Let $a = L/2$.

$$\left(\frac{EI}{a^3} \begin{bmatrix} 4a^2 & -6a \\ -6a & 12 \end{bmatrix} + \frac{P}{a} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{Bmatrix} \theta_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \text{ Let } s = \frac{Pa^2}{2EI}$$

$$\begin{vmatrix} 2a^2 & -3a \\ -3a & 6+s \end{vmatrix} = (2s+3)a^2 = 0$$

$$s = \frac{3}{2}, \quad P_{cr} = -3 \frac{EI}{a^2} = -12 \frac{EI}{L^2}$$

(c)

$$\left(\frac{EI}{L^3} \begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix} + \begin{bmatrix} \frac{2EI}{L^3} & 0 \\ 0 & 0 \end{bmatrix} + \frac{P}{L} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Let } s = PL^2/2EI$$

$$\begin{vmatrix} 7+s & -3L \\ -3L & 2L^2 \end{vmatrix} = (2s+5)L^2 = 0 \quad s = -2.5 \quad P_{cr} = -5 \frac{EI}{L^2}$$

(d)

$$\left(\frac{EI}{L^3} \begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{EI}{L} \end{bmatrix} + \frac{P}{L} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Let } s = PL^3/EI$$

$$\begin{vmatrix} 12+s & -6L \\ -6L & 5L^2 \end{vmatrix} = (5s+24)L^2 = 0 \quad s = -4.8 \quad P_{cr} = -4.8 \frac{EI}{L^2}$$

(e) Set $v_2 = 0$ in equations of part (c).

$$\left[\frac{EI}{L^3} (4L^2) + 0 + \frac{P}{L} (0) \right] \theta_2 = 0 \quad \text{No buckling: } \theta_2 = 0 \text{ is the only solution.}$$

(f) Set $\theta_2 = 0$ in equations of part (d).

$$\left[\frac{EI}{L^3} (12) + 0 + \frac{P}{L} (1) \right] v_2 = 0, \quad P_{cr} = -12 \frac{EI}{L^2}$$

18.5-8

(a) In dynamics, we choose as masters the D_i for which M_{ii}/K_{ii} is large. By analogy, for buckling & stress-stiffening problem we choose as masters the D_i for which $K_{\sigma ii}/K_{ii}$ is large. Thus, slaves are apt to be rotational d.o.f. (see e.g. Eqs. 3.3-14 and 18.2-6).

$$(b) [\underline{I}] = \begin{bmatrix} 1 \\ 70.333/134.667 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.52228 \end{bmatrix}$$

$$[\underline{I}]^T([\underline{K}][\underline{I}]) = [\underline{I}]^T \begin{Bmatrix} 0.1554 \\ 0 \end{Bmatrix} = 0.1554$$

$$[\underline{I}]^T\{R\} = [\underline{I}]^T \begin{Bmatrix} 0.1 \\ 0 \end{Bmatrix} = 0.1, \quad v_2 = \frac{0.1}{0.1554} = 0.644 \text{ m}$$

$$(c) [\underline{I}] = \begin{bmatrix} 1 \\ 76.333/158.667 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.48109 \end{bmatrix}$$

$$[\underline{I}]^T([\underline{K}][\underline{I}]) = [\underline{I}]^T \begin{Bmatrix} 24.165 \\ 0 \end{Bmatrix} = 24.165$$

$$[\underline{I}]^T\{R\} = [\underline{I}]^T \begin{Bmatrix} 0.1 \\ 0 \end{Bmatrix} = 0.1, \quad v_2 = \frac{0.1}{24.165} = 0.00414 \text{ m}$$

(d) Keep v_2 as master.

$$[\underline{I}] = \begin{bmatrix} -(-6a)/4a^2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5/a \\ 1 \end{bmatrix} \quad \text{Apply to } [\underline{K}] \text{ and to } [\underline{K}_\sigma].$$

$$[\underline{I}]^T[\underline{K}][\underline{I}] = \frac{3EI}{a^3}, \quad [\underline{I}]^T[\underline{K}_\sigma][\underline{I}] = 36 \frac{P}{30a} = \frac{1.2P}{a}$$

$$\left(\frac{3EI}{a^3} + \frac{1.2P}{a} \right) v_2 = 0, \quad P_{cr} = -2.5 \frac{EI}{a^2} = -10 \frac{EI}{L^2}$$

(e) Same $[\underline{I}]$ as in part (d).

$$[\underline{I}]^T[\underline{K}_\sigma][\underline{I}] = \frac{14.25P}{12a}$$

$$\left(\frac{3EI}{a^3} + \frac{14.25P}{12a} \right) v_2 = 0, \quad P_{cr} = -2.526 \frac{EI}{a^2} = -10.11 \frac{EI}{L^2}$$

18.5-9

$$\left(\frac{AE}{L} + \frac{AE}{\alpha L}\right) u_B = P, \quad u_B = \frac{\alpha}{1+\alpha} \frac{PL}{AE}$$

$$P_{AB} = \frac{AE}{L} u_B = \frac{\alpha P}{1+\alpha}, \quad P_{BC} = -\frac{AE}{\alpha L} u_B = -\frac{P}{1+\alpha}$$

$$(k_{AB} + k_{BC} + k_{\sigma AB} + k_{\sigma BC}) \theta_B = 0$$

$$\left[\left(\frac{4EI}{L} + \frac{4EI}{(1+\alpha)L} \right) + \underbrace{\left(\frac{2(\alpha P)L}{15} + \frac{-2P(\alpha L)}{15} \right)}_{=0} \right] \theta_B = 0$$

The only solution is $\theta_B = 0$: no buckling.

18.5-10

$$\left[\left(\frac{4EI}{1.3a} + \frac{4EI}{a} \right) + \frac{2}{15} (-1.3aP \cos \beta - aP \sin \beta) \right] \theta_B = 0$$

$$(1.3 \cos \beta + \sin \beta) P = \frac{30EI}{a^2} \left(\frac{1}{1.3} + 1 \right) = 53.077 \frac{EI}{a^2}$$

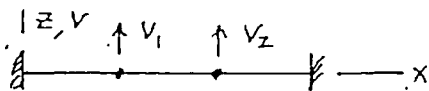
$$P = 53.077 \frac{EI}{a^2} \frac{1}{1.3 \cos \beta + \sin \beta} \quad (A)$$

Want to minimize $1.3 \cos \beta + \sin \beta$.

$$-1.3 \sin \beta + \cos \beta = 0, \quad \beta = \arctan \frac{1}{1.3} = 37.6^\circ$$

By using adjacent angles in Eq. (A), we see that $\beta = 37.6^\circ$ provides a min. of P , not a max.

18.6-1



(a) After discarding fixed d.o.f. at the ends,
 $[K_0]\{D\} = \{R\}$ becomes ($g = \text{accel. of gravity}$)

$$\frac{T}{L} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} -mg \\ -mg \end{Bmatrix}, \quad \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = -\frac{mgL}{T} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

(b) With $[K]$ and $\{R\}$ both zero, Eq. 14.6-1 becomes

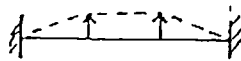
$$\left(\frac{T}{L} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - m\omega^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let $s = m\omega^2 L / T$

$$\begin{vmatrix} 2-s & -1 \\ -1 & 2-s \end{vmatrix} = s^2 - 4s + 3 = 0, \quad \begin{cases} s = 1 \\ s = 3 \end{cases}$$

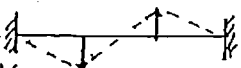
For $s = 1$, $\omega^2 = \frac{T}{mL}$

$$\begin{bmatrix} 2-1 & -1 \\ -1 & 2-1 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad v_1 = v_2$$



For $s = 3$, $\omega^2 = \frac{3T}{mL}$

$$\begin{bmatrix} 2-3 & -1 \\ -1 & 2-3 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad v_1 = -v_2$$



18.6-2

(a)

$$\frac{T}{L} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} -2mg \\ -mg \end{Bmatrix}, \quad \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = -\frac{mgL}{3T} \begin{Bmatrix} 5 \\ 4 \end{Bmatrix}$$

$$(b) \left(\frac{T}{L} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \omega^2 \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \right) \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

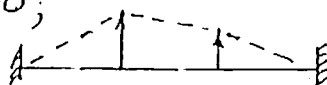
Let $s = m\omega^2 L / T$

$$\begin{vmatrix} 2-2s & -1 \\ -1 & 2-s \end{vmatrix} = 2s^2 - 6s + 3 = 0, \quad \begin{cases} s = 0.634 \\ s = 2.366 \end{cases}$$

For $s = 0.634$, $\omega^2 = 0.634 T / mL$

$$[2 - 2(0.634)]v_1 - v_2 = 0;$$

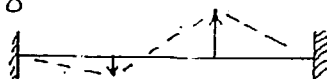
$$v_1 = 1.366 v_2$$



For $s = 2.366$, $\omega^2 = 2.366 T / mL$

$$[2 - 2(2.366)]v_1 - v_2 = 0$$

$$v_1 = -0.366 v_2$$



18.6-3

(a) Let $L = 2a$, $[K] = [0]$ in Eq. 18.6-2.

$$\left(\frac{TL}{30} \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} - \omega^2 \frac{\rho L^3}{420} \begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix} \right) \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Let } s = \omega^2 \rho L^2 / 14T$$

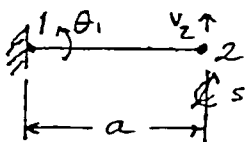
$$\begin{vmatrix} 4-4s & -1+3s \\ -1+3s & 4-4s \end{vmatrix} = 0 \quad \begin{cases} s = \frac{5}{7}, \omega_1^2 = 10 \frac{T}{\rho L^2} = 2.5 \frac{T}{\rho a^2} \\ s = 3, \omega_3^2 = 42 \frac{T}{\rho L^2} = 10.5 \frac{T}{\rho a^2} \end{cases}$$

$$\text{Mode 1: } (-1+3\frac{5}{7})\theta_1 + (4-4\frac{5}{7})\theta_2 = 0$$

$$\theta_1 = -\theta_2$$

$$\text{Mode 2: } (-1+9)\theta_1 + (4-12)\theta_2 = 0$$

$$\theta_1 = \theta_2$$

(b)  Thus we exclude mode 2; we obtain only mode 1 and mode 3.

$$\left(\frac{T}{30a} \begin{bmatrix} 4a^2 & -3a \\ -3a & 36 \end{bmatrix} - \frac{\rho \omega^2}{420} \begin{bmatrix} 4a^3 & 13a^2 \\ 13a^2 & 156a \end{bmatrix} \right) \begin{Bmatrix} \theta_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Let } s = 30\rho\omega^2 a^2 / 420T$$

$$\begin{vmatrix} 4a^2-4sa^2 & -3a-13sa \\ -3a-13sa & 36-156s \end{vmatrix} = (455s^2 - 846s + 135)a^2 = 0$$

$$\text{Mode 1: } s = 0.17629, \omega_1^2 = 2.468 \frac{T}{\rho a^2}$$

$$4a^2(1-0.17629)\theta_1 - a[3+13(0.17629)]v_2 = 0$$

$$\theta_1 = 1.606 \frac{v_2}{a}$$

$$\text{Mode 3: } s = 1.68305, \omega_3^2 = 23.56 \frac{T}{\rho a^2}$$

$$15\theta_1 - a[3+13(1.68305)]v_2 = 0$$

$$\theta_1 = -9.106 \frac{v_2}{a}$$

18.6-4

(a) Cubic [M]

18.6-3(a): $[K_r]$ is null if d.o.f. are θ_1 and θ_2 only; no solution.

$$18.6-3(b): \left(\frac{T}{a} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \frac{\rho \omega^2}{420} \begin{bmatrix} 4a^3 & 13a^2 \\ 13a^2 & 156a \end{bmatrix} \right) \begin{Bmatrix} \theta_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\text{Let } s = \rho \omega^2 a^2 / 420 T$$

$$\begin{vmatrix} -4sa^2 & -13sa \\ -13sa & 1-156s \end{vmatrix} = (455s-4)sa^2 = 0$$

$$s = 0.0087912, \quad \omega^2 = 3.692 \frac{T}{\rho a^2}$$

$$0 = -4sa^2 \theta_1 - 13sa v_2, \quad \theta_1 = -\frac{13}{4} \frac{v_2}{a}$$

This result is physically meaningless.

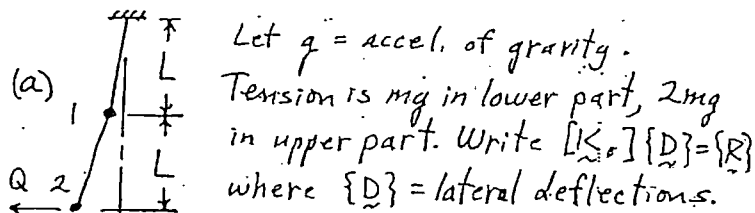
(b) Lumped [M]

18.6-3(a): As before, no solution.

$$18.6-3(b): \left(\frac{T}{a} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \omega^2 \frac{\rho a}{2} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{Bmatrix} \theta_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\left(\frac{T}{a} - \omega^2 \frac{\rho a}{2} \right) v_2 = 0, \quad \omega^2 = 2 \frac{T}{\rho a^2}$$

18.6-5



$$\frac{mg}{L} \begin{bmatrix} 2+1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ Q \end{Bmatrix}, \quad \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \frac{QL}{2mg} \begin{Bmatrix} 1 \\ 3 \end{Bmatrix}$$

(b) With $[K]=\{R\}=\{0\}$, Eq. 18.6-1 becomes

$$\left(\frac{mg}{L} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2 m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \quad \text{Let } s = \frac{\omega^2 L}{g}$$

$$\begin{vmatrix} 3-s & -1 \\ -1 & 1-s \end{vmatrix} = s^2 - 4s + 2 = 0 \quad \begin{cases} s = 0.586 \\ s = 3.414 \end{cases}$$

For $s = 0.586$, $\omega^2 = 0.586 g/L$

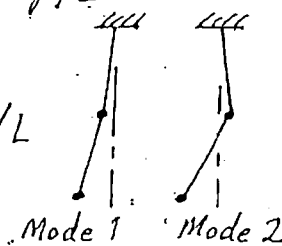
$$-v_1 + (1-0.586)v_2 = 0$$

$$v_1 = 0.414 v_2$$

For $s = 3.414$, $\omega^2 = 3.414 g/L$

$$-v_1 + (1-3.414)v_2 = 0$$

$$v_1 = -2.414 v_2$$



18.6-6

Use Eq. 18.6-2 for this problem.

$$\left(\frac{300}{1^3} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} - \omega^2 \frac{2100(0.0002)}{420} \begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix} + \frac{T}{30} \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \right) \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Set $\theta_2 = -\theta_1$; then

$$\left(600 - \frac{7\omega^2}{1000} + \frac{T}{6} \right) \theta_1 = 0$$

$$(a) T = 0; \quad \omega = \left[\frac{6(10)^5}{7} \right]^{1/2} = 293 \text{ rad/s}$$

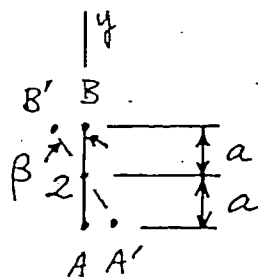
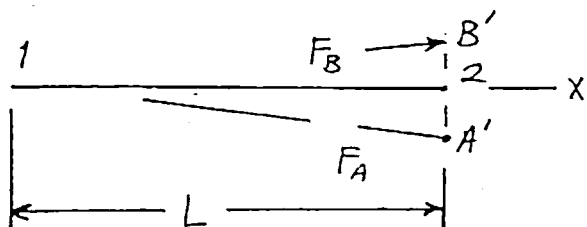
$$(b) \omega^2 = 347^2; \quad T = 6 \left[\frac{7(347)^2}{1000} - 600 \right] = 1457 \text{ N}$$

$$(c) T = -1200; \quad \omega = \left[\frac{1000}{7} \left(600 + \frac{-1200}{6} \right) \right]^{1/2} = 239 \frac{\text{rad}}{\text{s}}$$

18.6-7

Imagine a small rotation of the bars of length a about the x axis.

View along the y axis:



$$\alpha = \frac{\beta a}{L} \quad (\text{a small angle})$$

$$F_A = F_B = mL\Omega^2$$

Torque about x axis due to F_A and F_B is

$$T = 2(F_A \alpha) a = 2mL\Omega^2 \alpha a = 2mL\Omega^2 \frac{\beta a^2}{L} = 2m\Omega^2 \beta a^2$$

Torque generated due to twist of bar 1-2 is

$$\text{rotation} = \frac{TL}{GK} \quad \text{so} \quad T = \frac{GK\beta}{L}$$

Equate torques

$$\frac{GK\beta}{L} = 2m\Omega^2 \beta a^2 \quad \text{hence} \quad \Omega^2 = \frac{GK}{2mLa^2}$$